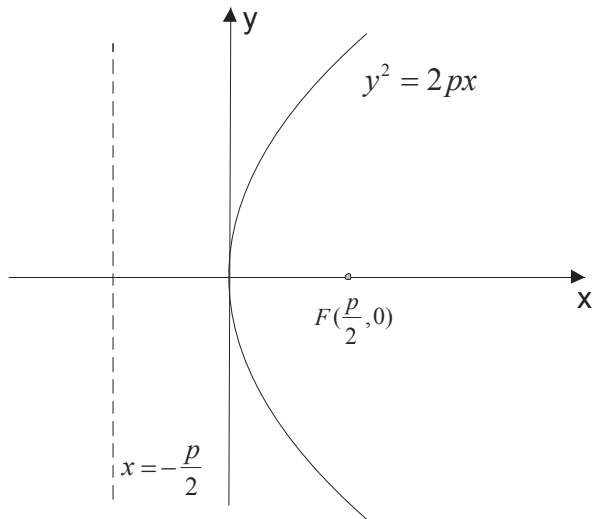


PARABOLA

Parabola is the set of points in the plane with attribute that distance of each point from a constant point (focus) is the same as distance from the point of a permanent line (directrix).



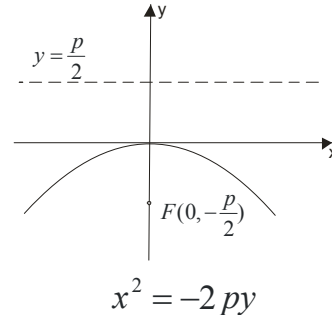
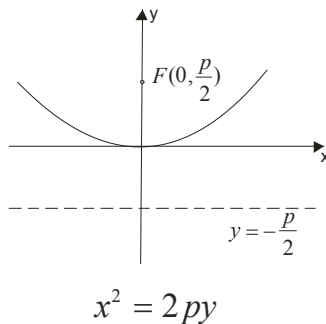
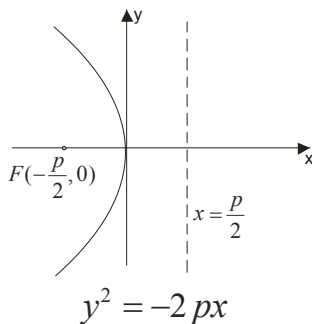
$F(\frac{p}{2}, 0)$ - this is the focus point of the parabola

Line $x = -\frac{p}{2}$ is the directrix of the parabola, $(x + \frac{p}{2} = 0)$

Distance from point $F(\frac{p}{2}, 0)$ to directrix we will marked with p (the parameter of the parabola).

The equation is: $y^2 = 2px$

Of course, this parabola is the most studied, but here is some other parabolas:

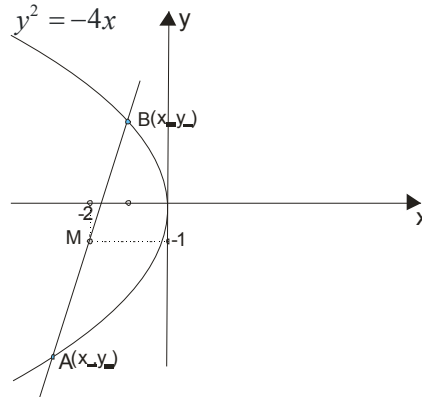


Example 1.

We have parabola $y^2 = -4x$. Through its point $M(-2, -1)$ set a chord that is the point halved.

Solution:

Draw a picture to analyze the problem:



Point M in mid-long AB and must be :

$$\frac{x_1 + x_2}{2} = -2 \rightarrow x_1 + x_2 = -4$$

$$\frac{y_1 + y_2}{2} = -1 \rightarrow y_1 + y_2 = -2$$

Points A and B belong to the parabola, and their coordinates can be changed instead x and y in equation:

$$A(x_1, y_1) \in y^2 = -4x \rightarrow y_1^2 = -4x_1$$

$$B(x_2, y_2) \in y^2 = -4x \rightarrow y_2^2 = -4x_2$$

In this way we get 4 equations with 4 unknowns. We ask for the coordinates of points A and B, but smarter is to find the direction of line which passes through AB .

$$k = \frac{y_2 - y_1}{x_2 - x_1}$$

$$y_1^2 = -4x_1$$

$$y_2^2 = -4x_2$$

$$y_2^2 - y_1^2 = -4x_2 - (-4x_1)$$

$$(y_2 - y_1)(y_2 + y_1) = -4x_2 + 4x_1 \quad \text{we use that } y_2 + y_1 = -2$$

$$(y_2 - y_1)(-2) = -4(x_2 - x_1)$$

$$\frac{y_2 - y_1}{x_2 - x_1} = 2$$

Now the equation of line through a point $M(-2, -1)$

$$y - y_0 = k(x - x_0)$$

$$y - (-1) = 2(x - (-2))$$

$$y + 1 = 2x + 4$$

$$y = 2x + 3$$

Line and parabola

Similarly as in the circle, ellipse and hyperbola, to determine the mutual position of line and parabola, solve the system of equations:

$$y = kx + n \quad \text{and} \quad y^2 = 2px$$

- If the system has no solution, then the line and the parabola is not cut, that is $p < 2kn$
- If the system has two solutions, then line cut parabola in two points $p > 2kn$
- If the system has one solution, line is tangent, and satisfies the **contact condition**: $p = 2kn$

Note;

If we seek an tangent line at a given point (x_0, y_0) which belongs to the parabola, we have formula:

$$y \cdot y_0 = p(x + x_0)$$

Example 2.

Find a point on $y^2 = 16x$ whose tangent line is tilted to an angle of 135 degrees (135°) to the x-axis

Solution:

Mark that point (x_0, y_0) . Tangent will be $y \cdot y_0 = p(x + x_0)$. From parabola, we have:

$$2p = 16$$

$$p = 8$$

$$y \cdot y_0 = 8(x + x_0)$$

$$y \cdot y_0 = 8x + 8x_0$$

$$y = \frac{8}{y_0}x + \frac{8x_0}{y_0}$$

$$k = \frac{8}{y_0}$$

We found line direction, and how:

$$k = \operatorname{tg} \alpha$$

$$k = \operatorname{tg} 135^\circ$$

$$k = -1$$

$$k = \frac{8}{y_0}$$

Then is:

$$-1 = \frac{8}{y_0} \rightarrow y_0 = -8$$

Substituting this value in the equation of parabola ,we have

$$y^2 = 16x$$

$$(-8)^2 = 16x$$

$$16x = 64$$

$$x = 4$$

Therefore, the requested point is (4, -8)

Example 3.

Write the equation of tangent parabola $y^2 = 12x$ if it is parallel with line $3x - y - 4 = 0$

Solution:

How is our line parallel to the given, they have the same k (condition parallels)

$$3x - y - 4 = 0$$

$$y = 3x - 4$$

$$k = 3$$

For now, we have $y = 3x + n$, and n will be found by using **contact condition:** $p = 2kn$

From parabola is $y^2 = 12x \rightarrow 2p = 12 \rightarrow p = 6$

$$p = 2kn$$

$$6 = 2 \cdot 3n$$

$$6 = 6n$$

$$n = 1$$

Solution is: $y = 3x + 1$

Example 4.

Write the equation of the common tangent line for $y^2 = 4x$ and $x^2 + y^2 - 2x - 9 = 0$

Solution:

$$x^2 + y^2 - 2x - 9 = 0$$

$$x^2 - 2x + y^2 - 9 = 0$$

$$\underline{x^2 - 2x + 1} - 1 + y^2 - 9 = 0$$

$$(x-1)^2 + y^2 = 10 \rightarrow p = 1, q = 0, r^2 = 10$$

Search tangent: $y = kx + n$.

It must satisfy contact condition with parabola and with circle.

From parabola we have $y^2 = 4x$, then $2p = 4$, so $p = 2$

$$p = 2kn$$

$$2 = 2kn$$

$$\boxed{kn = 1}$$

For circle:

$$r^2(k^2 + 1) = (kp - q + n)^2$$

$$10(k^2 + 1) = (1 \cdot k + n)^2$$

$$\boxed{10(k^2 + 1) = (k + n)^2}$$

We have two equations with two unknowns, solve the system:

$$\boxed{10(k^2 + 1) = (k + n)^2}$$

$$\boxed{kn = 1}$$

$$n = \frac{1}{k}$$

$$10(k^2 + 1) = \left(k + \frac{1}{k}\right)^2$$

$$10(k^2 + 1) = \left(\frac{k^2 + 1}{k}\right)^2$$

$$10(k^2 + 1) = \frac{(k^2 + 1)^2}{k^2}$$

$$10k^2 = k^2 + 1$$

$$9k^2 = 1$$

$$k^2 = \frac{1}{9} \rightarrow k_1 = \frac{1}{3}, k_2 = -\frac{1}{3}$$

Let's go back to find n:

$$kn = 1$$

$$k_1 = \frac{1}{3} \rightarrow n_1 = 3$$

$$k_2 = -\frac{1}{3} \rightarrow n_2 = -3$$

Tangent line equations are:

$$t_1: y = \frac{1}{3}x + 3$$

$$t_2: y = -\frac{1}{3}x - 3$$

Why do we appear two solutions?

If we draw the problem, we see that it is obvious ..

