

Trigonometric form of a complex number (polar form)

To remind you:

Complex number is : $z = x + yi$

x is a real part, y is imaginary part of a complex number, and i is imaginary unit : $i = \sqrt{-1}$ ($i^2 = -1$)

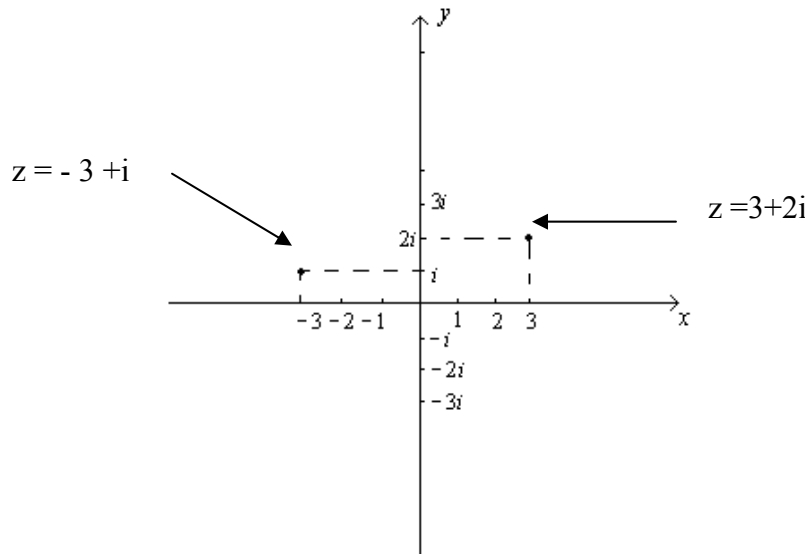
Two complex number $z_1 = x_1 + y_1i$ and $z_2 = x_2 + y_2i$ are equal if $x_1 = x_2$ and $y_1 = y_2$

For $z = x + yi$, number $\bar{z} = x - yi$ is a complex conjugate number.

Module of complex number $z = x + yi$ is $|z| = \sqrt{x^2 + y^2}$

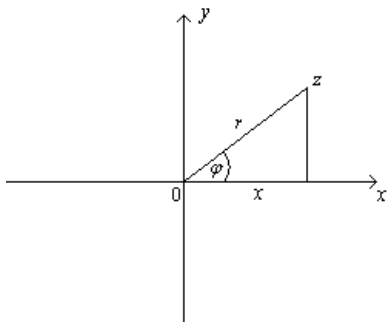
Complex numbers are presented in the complex plane, where x -line is real line and y -line is imaginary line:

Example:



If we have complex number $z = x + yi$ then the real part can write as: $x = r \cos \varphi$, and imaginary $y = r \sin \varphi$

This we can see from the picture:



$$\sin \varphi = \frac{y}{r} \Rightarrow y = r \sin \varphi$$

$$\cos \varphi = \frac{x}{r} \Rightarrow x = r \cos \varphi$$

$$\operatorname{tg} \varphi = \frac{y}{x} \Rightarrow \varphi = \operatorname{arctg} \frac{y}{x}$$

Therefore, the complex number is:

$$z = r \cos \varphi + r \sin \varphi i, \text{ or}$$

$$z = r(\cos \varphi + i \sin \varphi)$$

This form is called trigonometric. Here, r is module $r = \sqrt{x^2 + y^2}$; angle φ is the argument of the complex number. How are $\sin x$ and $\cos x$ periodic functions complex number we can write as:

$$z = r(\cos(\varphi + 2k\pi) + i \sin(\varphi + 2k\pi))$$

$$k \in \mathbb{Z}$$

Example: Make the following complex numbers in a trigonometrical form:

a) $z = 1 + i$

b) $z = 1 + i\sqrt{3}$

c) $z = -1$

d) $z = i$

Solution:

a) $z = 1 + i$

What we do?

First, determine x and y , then find $r = \sqrt{x^2 + y^2}$, then $\operatorname{tg} \varphi = \frac{y}{x}$ and replace that to trigonometriski form:

$$z = r(\cos \varphi + i \sin \varphi)$$

So: $x = 1, y = 1 \quad r = \sqrt{1^2 + 1^2} = \sqrt{2}$

$$\operatorname{tg} \varphi = \frac{y}{x}$$

$$\operatorname{tg} \varphi = \frac{1}{1}$$

$$\operatorname{tg} \varphi = 1 \Rightarrow \varphi = 45^\circ = \frac{\pi}{4}$$

$$z = r(\cos \varphi + i \sin \varphi)$$

$$z = \sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}) \text{ solution}$$

$$\text{b) } z = 1 + i\sqrt{3}$$

$$x = 1$$

$$y = \sqrt{3} \Rightarrow r = \sqrt{x^2 + y^2} = \sqrt{1+3} = \sqrt{4} = 2$$

$$\operatorname{tg} \varphi = \frac{y}{x} = \frac{\sqrt{3}}{1}$$

$$\operatorname{tg} \varphi = \sqrt{3}$$

$$\varphi = 60^\circ = \frac{\pi}{3}$$

$$z = r(\cos \varphi + i \sin \varphi)$$

$$z = 2\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$$

$$\text{c) } z = -1 \quad \text{Take heed: This can record as: } z = -1 + 0i$$

$$x = -1, y = 0$$

$$r = \sqrt{(-1)^2 + 0^2} = 1$$

$$\text{So : } \operatorname{tg} \varphi = \frac{y}{x} = \frac{0}{-1} = 0$$

$$z = r(\cos \varphi + i \sin \varphi)$$

$$\varphi = 180^\circ$$

$$z = 1 \cdot (\cos \pi + i \sin \pi)$$

$$\varphi = \pi$$

$$z = \cos \pi + i \sin \pi$$

$$\text{d) } z = i \text{ or } z = 0 + 1i \Rightarrow x = 0, y = 1$$

$$r = \sqrt{0^2 + 1^2} = 1$$

$$z = r(\cos \varphi + i \sin \varphi)$$

$$\operatorname{tg} \varphi = \frac{1}{0} = \infty$$

$$z = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$$

$$\varphi = \frac{\pi}{2}$$

Often, in solving tasks we use **Euler formula**:

$$e^{xi} = \cos x + i \sin x$$

Example: Write numbers through Euler formula.

a) 1

b) i

c) -2

Solution:

Tip: Here always add periodicity!

a) $z = 1 \longrightarrow x = 1, y = 0$

$$r = \sqrt{1^2 + 0^2} = 1$$

$$\operatorname{tg} \varphi = \frac{y}{x} = 0 \Rightarrow \varphi = 0^\circ$$

$$z = r(\cos(\varphi + 2k\pi) + i \sin(\varphi + 2k\pi))$$

$$z = 1 \cdot (\cos(0 + 2k\pi) + i \sin(0 + 2k\pi))$$

So: $1 = \cos 2k\pi + i \sin 2k\pi$, or

$$1 = e^{2k\pi i}$$

$$k \in \mathbb{Z}$$

b) $z = i \Rightarrow z = 0 + 1i \Rightarrow x = 0, y = 1$

$$r = \sqrt{x^2 + y^2} = \sqrt{0^2 + 1^2} = \sqrt{1} = 1$$

$$\operatorname{tg} \varphi = \frac{y}{x} = \frac{1}{0} = \infty \Rightarrow \varphi = \frac{\pi}{2}$$

$$z = r(\cos(\varphi + 2k\pi) + i \sin(\varphi + 2k\pi))$$

$$z = \cos\left(\frac{\pi}{2} + 2k\pi\right) + i \sin\left(\frac{\pi}{2} + 2k\pi\right)$$

So: $i = \cos\left(\frac{\pi}{2} + 2k\pi\right) + i \sin\left(\frac{\pi}{2} + 2k\pi\right)$

$$i = e^{\left(\frac{\pi}{2} + 2k\pi\right)i}$$

$$k \in \mathbb{Z}$$

c) $z = -2 = 2 \cdot (-1) =$

-1 \longrightarrow we find in the previous example:

$$-1 = \cos(\pi + 2k\pi) + i \sin(\pi + 2k\pi)$$

$$-2 = 2[\cos(\pi + 2k\pi) + i \sin(\pi + 2k\pi)]$$

So:

$$-2 = 2e^{(\pi+2k\pi)i}$$

$$-2 = 2e^{(2k+1)\pi i}$$

$$k \in \mathbb{Z}$$

Professors often like to ask the children to find values i^i

When you know Euler's record, it is not difficult.

In a previous example, we find:

$$i = e^{\frac{\pi}{2} + 2k\pi i}$$

$$k \in \mathbb{Z}$$

Then: $i^i = (e^{\frac{\pi}{2} + 2k\pi i})^i$

$$i^i = e^{\frac{\pi}{2} + 2k\pi i^2}$$

$$i^i = e^{-\frac{\pi}{2} + 2k\pi}$$

$$k \in \mathbb{Z}$$

Multiplication of complex numbers in trigonometric form

If we have two complex number in the trigonometric form:

$$z_1 = r_1(\cos \varphi_1 + i \sin \varphi_1)$$

then:

$$z_2 = r_2(\cos \varphi_2 + i \sin \varphi_2)$$

$$z_1 \cdot z_2 = r_1 \cdot r_2 [\cos(\varphi_1 + \varphi_2) + i \sin(\varphi_1 + \varphi_2)]$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\varphi_1 - \varphi_2) + i \sin(\varphi_1 - \varphi_2)]$$

Example: We have

$$z_1 = 4\sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})$$

$$z_2 = \frac{\sqrt{2}}{2}(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4})$$

find: a) $z_1 \cdot z_2$ b) $\frac{z_1}{z_2}$

Solution:

a)

$$\begin{aligned}z_1 \cdot z_2 &= 4\sqrt{2} \cdot \frac{\sqrt{2}}{2} \left[\cos\left(\frac{\pi}{4} + \frac{3\pi}{4}\right) + i \sin\left(\frac{\pi}{4} + \frac{3\pi}{4}\right) \right] \\&= 4[\cos \pi + i \sin \pi] \\&= 4[-1 + 0] \\&= -4\end{aligned}$$

b)

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{4\sqrt{2}}{\sqrt{2}} \left[\cos\left(\frac{\pi}{4} - \frac{3\pi}{4}\right) + i \sin\left(\frac{\pi}{4} - \frac{3\pi}{4}\right) \right] \\&= 8 \left[\cos\left(-\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right) \right] \\&= 8[0 - i \cdot 1] \\&= -8i\end{aligned}$$

De Moivre's formula

If we have $z = r(\cos \varphi + i \sin \varphi)$ then $z^n = r^n(\cos n\varphi + i \sin n\varphi)$

If the complex has a number has a module 1, ie. if $r = 1$ then:

$$z = \cos \varphi + i \sin \varphi$$

$$z^n = \cos n\varphi + i \sin n\varphi \rightarrow \text{De Moivre's formula}$$

Example: a) Find z^6 if $z = 2\left(\cos \frac{\pi}{18} + i \sin \frac{\pi}{18}\right)$

b) Find z^{20} if $z = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

Solution:

a)

$$z = 2\left(\cos \frac{\pi}{18} + i \sin \frac{\pi}{18}\right)$$

$$z^6 = 2^6 \left(\cos 6 \cdot \frac{\pi}{18} + i \sin 6 \cdot \frac{\pi}{18}\right)$$

$$z^6 = 2^6 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$$

$$z^6 = 64\left(\frac{1}{2} + i \frac{\sqrt{3}}{2}\right) = 32(1 + i\sqrt{3})$$

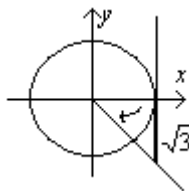
$$\text{b) } z = \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

Here we first must move complex number in the trigonometric form.

$$\left. \begin{array}{l} x = \frac{1}{2} \\ y = -\frac{\sqrt{3}}{2} \end{array} \right\} \Rightarrow r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

$$\operatorname{tg} \varphi = \frac{y}{x}$$

$$\operatorname{tg} \varphi = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} \Rightarrow \operatorname{tg} \varphi = -\sqrt{3} \Rightarrow \varphi = -60^\circ = -\frac{\pi}{3}$$



$$z = r(\cos \varphi + i \sin \varphi)$$

$$z = 1\left(\cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right)\right)$$

$$z = \cos \frac{\pi}{3} - i \sin \frac{\pi}{3}$$

$$z^{20} = \cos \frac{20\pi}{3} - i \sin \frac{20\pi}{3} \rightarrow \frac{20\pi}{3} = \frac{18\pi}{3} + \frac{2\pi}{3} = \frac{2\pi}{3}$$

$$z^{20} = \cos \frac{2\pi}{3} - i \sin \frac{2\pi}{3}$$

$$z^{20} = -\frac{1}{2} - i \frac{\sqrt{3}}{2}$$

$$z^{20} = -\frac{1}{2}(1 + i\sqrt{3})$$

Root of complex numbers

If we have $z = r(\cos \varphi + i \sin \varphi)$ then $\sqrt[n]{z} = w = \sqrt[n]{r} \left(\cos \frac{\varphi + 2k\pi}{n} + i \sin \frac{\varphi + 2k\pi}{n} \right) \quad k = 0, 1, 2, \dots, n-1$

All values n - the root of z , is on circle with radius $\sqrt[n]{r}$.

Example:

Calculate:

a) $\sqrt[3]{i}$

b) $\sqrt[6]{-1}$

Solution:

a) As we already see:

$$i = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$$

$$\text{Then : } \sqrt[3]{i} = \cos \frac{\frac{\pi}{2} + 2k\pi}{3} + i \sin \frac{\frac{\pi}{2} + 2k\pi}{3} \quad k = 0,1,2$$

For k=0

$$w_0 = \cos \frac{\frac{\pi}{2} + 0}{3} + i \sin \frac{\frac{\pi}{2} + 0}{3}$$

$$w_0 = \cos \frac{\pi}{6} + i \sin \frac{\pi}{6}$$

$$w_0 = \frac{\sqrt{3}}{2} + i \frac{1}{2}$$

For k=1

$$w_1 = \cos \frac{\frac{\pi}{2} + 2\pi}{3} + i \sin \frac{\frac{\pi}{2} + 2\pi}{3}$$

$$w_1 = \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}$$

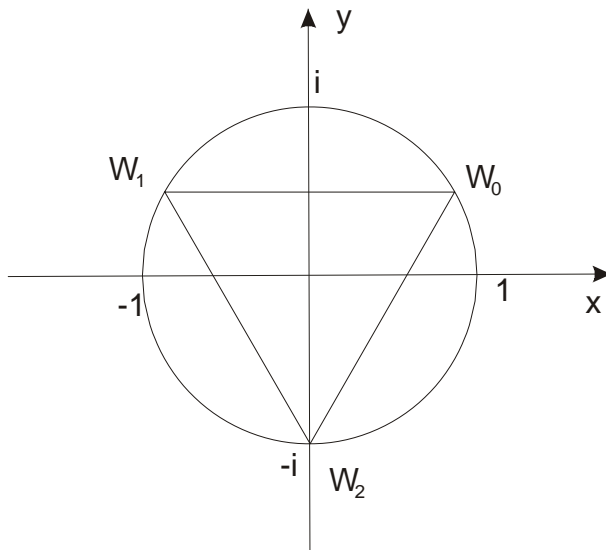
$$w_1 = -\frac{\sqrt{3}}{2} + i \frac{1}{2}$$

For k=2

$$w_2 = \cos \frac{\frac{\pi}{2} + 4\pi}{3} + i \sin \frac{\frac{\pi}{2} + 4\pi}{3}$$

$$w_2 = \cos \frac{9\pi}{6} + i \sin \frac{9\pi}{6}$$

$$w_2 = -i$$



b) $-1 = \cos \pi + i \sin \pi \quad r = 1, \varphi = \pi$
 $\sqrt[6]{-1} = \cos \frac{\pi + 2k\pi}{6} + i \sin \frac{\pi + 2k\pi}{6}$
 $k = 0, 1, 2, 3, 4, 5$

For k=0

$$w_0 = \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} = \frac{\sqrt{3}}{2} + i \frac{1}{2}$$

For k=1

$$w_1 = \cos \frac{\pi + 2\pi}{6} + i \sin \frac{\pi + 2\pi}{6}$$

$$w_1 = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = i$$

For k=2

$$w_2 = \cos \frac{\pi + 4\pi}{6} + i \sin \frac{\pi + 4\pi}{6}$$

$$w_2 = \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + i \cdot \frac{1}{2}$$

For k=3

$$w_3 = \cos \frac{\pi + 6\pi}{6} + i \sin \frac{\pi + 6\pi}{6}$$

$$w_3 = \cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6} = -\frac{\sqrt{3}}{2} - i \frac{1}{2}$$

For k=4

$$w_4 = \cos \frac{\pi + 8\pi}{6} + i \sin \frac{\pi + 8\pi}{6}$$

$$w_4 = \cos \frac{9\pi}{6} + i \sin \frac{9\pi}{6} = -i$$

For k=5

$$w_5 = \cos \frac{\pi + 10\pi}{6} + i \sin \frac{\pi + 10\pi}{6}$$

$$w_5 = \cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6} = \frac{\sqrt{3}}{2} - i \frac{1}{2}$$

