

## Mathematical induction

**Mathematical induction** is a method of mathematical proof typically used to establish that a given statement is true of all natural numbers.

The simplest and most common form of mathematical induction proves that a statement involving a natural number  $n$  holds for all values of  $n$ .

**The principle of mathematical induction is as follows:**

**For the statement  $T(n)$ ,  $n \in N$  is:**

- 1)  $T(1)$  is true [ or in some cases  $T(0)$  is true ]
- 2)  $T(n) \Rightarrow T(n+1)$  is true for  $\forall n = 1, 2, \dots$  then, the statement  $T(n)$  is true for  $\forall n \in N$

**Practically, we will do:**

- 1) Check whether the formula is correct for  $n = 1$
- 2) Suppose that the formula is correct for  $n = k$  [**induction hypothesis**]
- 3) Argues that the formula is correct for  $n = k + 1$

### EXAMPLES:

- 1) **Prove that the statement:**  $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$  **holds for all natural numbers  $n$ .**

**Proof:**

- i) Check whether the formula is correct for  $n = 1$  (instead  $n$ , we put 1)

$$1 = \frac{1(1+1)}{2} \Rightarrow 1 = 1 \quad \text{correct}$$

- ii) **Induction hypothesis:** suppose that the formula is correct for  $n = k$  (instead  $n$ , we put  $k$ )

$$1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2}$$

iii) Argues that the formula is correct for  $n = k + 1$

First, see what you need to prove, in the initial formula replace  $n$  with  $k + 1$ , but **always on the left side write before- last member.**

$$1 + 2 + \dots + \overset{\uparrow}{k} + (k + 1) = \frac{(k + 1)(k + 1 + 1)}{2} \quad \text{or} \quad 1 + 2 + \dots + k + (k + 1) = \frac{(k + 1)(k + 2)}{2}$$

**before- last member**

**So, this should prove!**

Always **begin by hypothesis** that we have presumed that is always correct!

$$1 + 2 + 3 \dots + k = \frac{k(k + 1)}{2}$$

Stop a little, and compare the left side of the hypothesis and what should prove

$$1 + 2 + 3 \dots + k = \frac{k(k + 1)}{2}$$
$$1 + 2 + \dots + k + (k + 1) = \frac{(k + 1)(k + 2)}{2}$$

$\uparrow$

I see that the hypothesis is missing member  $(k + 1)$ . It is a **trick**, that on both sides of the hypothesis add expression  $(k + 1)$ .

$$1 + 2 + 3 \dots + k = \frac{k(k + 1)}{2}$$

$$1 + 2 + 3 \dots + k + (k + 1) = \frac{k(k + 1)}{2} + (k + 1)$$

When you calculate, the right side must be  $\frac{(k + 1)(k + 2)}{2}$

$$\text{So: } \frac{k(k + 1)}{2} + \frac{k + 1}{1} = \frac{(k + 1) + 2(k + 1)}{2} = \frac{(k + 1)(k + 2)}{2}$$

**This is a complete proof.**

2) Prove that the statement:  $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$  holds for all natural numbers  $n$ .

**Proof:**

i) Check whether the formula is correct for  $n = 1$

$$1^2 = \frac{1(1+1)(2 \cdot 1+1)}{6} \Rightarrow 1=1 \quad \text{correct}$$

ii) **Induction hypothesis:** suppose that the formula is correct for  $n = k$

$$1^2 + 2^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

iii) Argues that the formula is correct for  $n = k + 1$

Always first we see what we have to prove:

$$1^2 + 2^2 + \dots + k^2 + (k+1)^2 = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$$

or  $1^2 + 2^2 + \dots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

Head from hypothesis, and on both sides add  $(k+1)^2$

$$1^2 + 2^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

$$\underbrace{1^2 + 2^2 + \dots + k^2 + (k+1)^2}_{\text{left side is same}} = \underbrace{\frac{k(k+1)(2k+1)}{6} + (k+1)^2}_{\text{This must give : } \frac{(k+1)(k+2)(2k+3)}{6}}$$

left side is same                      This must give :  $\frac{(k+1)(k+2)(2k+3)}{6}$

So:

$$\frac{k(k+1)(2k+1)}{6} + \frac{(k+1)^2}{1} = \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

$$= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6}$$

$$= \frac{(k+1)[2k^2 + k + 6k + 6]}{6}$$

$$= \frac{(k+1)[2k^2 + 7k + 6]}{6}$$

Term  $2k^2 + 7k + 6$ , we'll separate on facts with  $ak^2 + bk + c = 0 \dots \dots \dots a(k - k_1)(k - k_2) = 0$

$$2k^2 + 7k + 6 = 0$$

$$k_{1,2} = \frac{-7 \pm 1}{4} \quad \text{So: } 2k^2 + 7k + 6 = 2\left(k + \frac{3}{2}\right)(k + 2) = (2k + 3)(k + 2)$$

$$k_1 = -\frac{3}{2}$$

$$k_2 = -2$$

Let's go back to the task:

$$\begin{aligned} \frac{k(k+1)(2k+1)}{6} + \frac{(k+1)^2}{1} &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\ &= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \\ &= \frac{(k+1)[2k^2 + k + 6k + 6]}{6} \\ &= \frac{(k+1)[2k^2 + 7k + 6]}{6} \\ &= \frac{(k+1)(2k+3)(k+2)}{6} \end{aligned}$$

And this we should prove!

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3) **Prove that the statement:**  $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \dots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$  **holds for all natural numbers  $n$ .**

**Proof:**

i)  $\frac{1}{(2 \cdot 1 - 1)(2 \cdot 1 + 1)} = \frac{1}{2 \cdot 1 + 1} \Rightarrow \frac{1}{3} = \frac{1}{3}$  correct

ii) **Induction hypothesis:**  $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \dots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}$

iii) Argues that the formula is correct for  $n = k + 1$

Prvo da vidimo šta treba da First to see what should be prove!

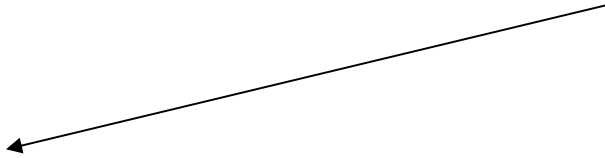
$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \dots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)} = \frac{k+1}{2k+3}$$

Proof will as usual start from the hypothesis:

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \dots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}$$

on both sides will add  $\frac{1}{(2k+1)(2k+3)}$

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \dots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)} = \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$$



$$\frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} = \frac{k(2k+3)+1}{(2k+1)(2k+3)} = \frac{2k^2+3k+1}{(2k+1)(2k+3)}$$

$$2k^2+3k+1=0 \quad 2k^2+3k+1=a(k-k_1)(k-k_2)$$

$$k_{1,2} = \frac{-3 \pm 1}{4} = 2 \left( k + \frac{1}{2} \right) (k+1)$$

$$k_1 = -\frac{1}{2} = (2k+1)(k+1)$$

$$k_2 = -1$$

Let's go back to the task:

$$\frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} = \frac{k(2k+3)+1}{(2k+1)(2k+3)} = \frac{2k^2+3k+1}{(2k+1)(2k+3)} = \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} = \frac{k+1}{2k+3}$$

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**4) Prove that  $5^{n-1} + 2^n$  is divisible by 3 for all natural numbers  $n$ .**

**Proof:**

i) for  $n = 1$   $5^{n-1} + 2^n = 5^{1-1} + 2^1 = 5^0 + 2 = 1 + 2 = 3$  correct, because  $3:3 = 1$

ii) Induction hypothesis :  $5^{k-1} + 2^k$  can be divided by 3

iii) To prove that  $5^{n-1} + 2^n$  can be divided with 3 for  $n = k + 1$

$$\begin{aligned}
5^{n-1} + 2^n &= 5^{k-1} + 2^{k+1} \\
&= 5^{k-1+1} + 2^k \cdot 2^1 \\
&= 5^{k-1} 5^1 + 2^k \cdot 2 \\
&= 5 \cdot 5^{k-1} + 2 \cdot 2^k
\end{aligned}$$

We use formula:  $a^{m+n} = a^m \cdot a^n$



Write as "trick":  $5 \cdot 5^{k-1} = 3 \cdot 5^{k-1} + 2 \cdot 5^{k-1}$

$$\begin{aligned}
&= 5 \cdot 5^{k-1} + 2 \cdot 2^k \\
&= 3 \cdot 5^{k-1} + 2 \cdot 5^{k-1} + 2 \cdot 2^k = 3 \cdot 5^{k-1} + 2(5^{k-1} + 2^k)
\end{aligned}$$

It is certainly divisible with 3 . Why?

$$\begin{array}{ccc}
3 \cdot 5^{k-1} + 2(5^{k-1} + 2^k) & & \\
\downarrow & \searrow & \\
\text{obviously, because of } 3 & & \text{is divisible by 3 because of our assumption that } 5^{k-1} + 2^k \text{ is divisible by 3}
\end{array}$$

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5) **Prove that  $6^{2n} + 3^{n+2} + 3^n$  is divisible by 11 for all natural numbers  $n$ .**

**Proof:**

i) for  $n=1$  is

$$\begin{aligned}
6^{2n} + 3^{n+2} + 3^n &= 6^2 + 3^3 + 3^1 \\
&= 36 + 27 + 3 \\
&= 66 = 6 \cdot 11 \\
&\text{correct}
\end{aligned}$$

ii) Induction hypothesis  $6^{2n} + 3^{n+2} + 3^n$  can be divided by 11

iii) Proof for  $n=k+1$

$$\begin{aligned}
&6^{2(k+1)} + 3^{k+1+2} + 3^{k+1} = \\
&6^{2k+1} + 3^{k+2+1} + 3^{k+1} = \\
&6^{2k} \cdot 6^2 + 3^{k+2} \cdot 3^1 + 3^k \cdot 3^1 = \quad (a^{m+n} = a^m \cdot a^n) \\
&36 \cdot 6^{2k} + 3 \cdot 3^{k+2} + 3 \cdot 3^k =
\end{aligned}$$

**Now, we need some ideas!**

As with  $6^{2k}$  we have 36, triples with  $3^{k+2}$  and  $3^k$  we will write as  $3 = 36 - 33$

So:

$$36 \cdot 6^{2k} + 36 \cdot 3^{k+2} - 33 \cdot 3^{k+2} + 36 \cdot 3^k - 33 \cdot 3^k =$$

$$= 36(6^{2k} + 3^{k+2} + 3^k) - 33(3^{k+2} + 3^k)$$

Term  $36(6^{2k} + 3^{k+2} + 3^k)$  is divisible by 11 because of assumptions and expression  $33(3^{k+2} + 3^k)$  because of number  $33 = 3 \cdot 11$

This is a complete proof.

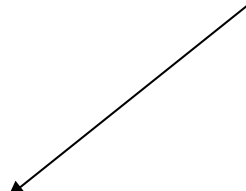
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6) Prove that for any natural number  $n > 1$  is  $\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} > \frac{13}{24}$

Solution:

i) For  $n=2$  is  $\frac{1}{3} + \frac{1}{4} = \frac{7}{12} = \frac{14}{24}$

$\frac{14}{24} > \frac{13}{24}$  correct



ii) Induction hypothesis: for  $n=k$   $\frac{1}{k+1} + \frac{1}{k+2} + \dots + \frac{1}{2k} > \frac{13}{24}$

iii) For  $n= k+ 1$ , we need to prove:  $\frac{1}{k+2} + \frac{1}{k+3} + \dots + \frac{1}{2k} + \frac{1}{2(k+1)} > \frac{13}{24}$

We must use a new “trick”!

Mark with:

$$S_k = \frac{1}{k+1} + \frac{1}{k+2} + \dots + \frac{1}{2k} \quad (S_k > \frac{13}{24}, \text{By assumption})$$

and

$$S_{k+1} = \frac{1}{k+2} + \frac{1}{k+3} + \dots + \frac{1}{2k} + \frac{1}{2k+1} + \frac{1}{2(k+1)}$$

Determine the difference  $S_{k+1} - S_k = ?$

$$\begin{aligned}
S_{k+1} - S_k &= \left( \frac{1}{k+2} + \frac{1}{k+3} + \dots + \frac{1}{2k+1} + \frac{1}{2(k+1)} \right) - \left( \frac{1}{k+1} + \frac{1}{k+2} + \dots + \frac{1}{2k} \right) \\
&= \frac{1}{k+2} + \frac{1}{k+3} + \dots + \frac{1}{2k} + \frac{1}{2k+1} + \frac{1}{2(k+1)} - \frac{1}{k+1} - \frac{1}{k+2} - \dots - \frac{1}{2k} \\
&= \frac{1}{2k+1} + \frac{1}{2(k+1)} - \frac{1}{k+1} \\
&= \frac{1 \cdot 2(k+1) + 1 \cdot (2k+1) - 2(k+1)}{(2k+1) \cdot 2 \cdot (k+1)} \\
&= \frac{2k+2+2k+1-4k-2}{2(2k+1)(k+1)} \\
&= \frac{1}{2(2k+1)(k+1)} > 0
\end{aligned}$$

This is certainly positive, because  $k > 0$ ,  $2k+1 > 0$  and  $k+1 > 0$

So:  $S_{k+1} - S_k > 0$

$$S_{k+1} > \underbrace{S_k}_{> \frac{13}{24}}$$

induction hypothesis

Then is:  $S_{k+1} > \frac{13}{24}$

**This is a complete proof !**

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**7) Prove that:  $2^n > n^2$  for everyone  $n \geq 5$**

**Proof:**

Proof begin with  $n = 5$ , because  $n \geq 5$

i)

$$\begin{aligned}
n=5 &\Rightarrow 2^5 > 5^2 \\
&36 > 25 \text{ correct}
\end{aligned}$$

ii) Induction hypothesis: for  $n=k$  is  $2^k > k^2$

iii) For  $n=k+1$ , we need to prove:  $2^{k+1} > (k+1)^2$

**Here we need new ideas!**

Watch the term :  $\left(1 + \frac{1}{n}\right)^2$

$$\left(1 + \frac{1}{n}\right)^2 = 1^2 + 2 \cdot 1 \cdot \frac{1}{n} + \left(\frac{1}{n}\right)^2 = 1 + \frac{2}{n} + \frac{1}{n^2}$$

$$\text{since } n \geq 5 \Rightarrow \frac{1}{n} \leq \frac{1}{5} \quad \text{and} \quad \frac{1}{n^2} \leq \frac{1}{25}$$

$$\text{then } \left(1 + \frac{1}{n}\right)^2 = 1 + \frac{2}{n} + \frac{1}{n^2} \leq 1 + \frac{2}{5} + \frac{1}{25}$$

$$1 + \frac{2}{5} + \frac{1}{25} = 1 + \frac{11}{25} < 2$$

$$\left(\frac{n+1}{n}\right)^2 < 2$$

$$\frac{(n+1)^2}{n^2} < 2$$

and then is  $\frac{(k+1)^2}{k^2} < 2$ , and hypothesis is  $2^k > k^2$

$$\left. \begin{array}{l} 2^k > k^2 \\ 2 > \frac{(k+1)^2}{k^2} \end{array} \right\} \text{multiply them! (left with left and right with right)}$$

$$2^k \cdot 2 > \frac{(k+1)^2}{k^2} \cdot k^2$$

$2^{k+1} > (k+1)^2 \rightarrow$  **and this is what we want to prove!**