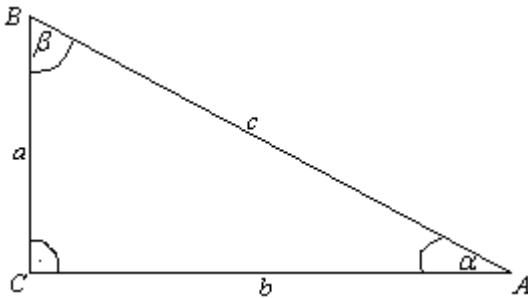


## Trigonometric functions of sharp angle

Trigonometry was originally an area of mathematics that deal with calculating unknown elements of the triangle with well-known. The name comes from two Greek words , TRIGONOS-meaning triangle and METRON-meaning measures. How do we define trigonometric functions?

Observe right-angled triangle ABC.



a, b → cathetus  
c → hypotenuse  
 $a^2 + b^2 = c^2$  → Pythagorean theorem

$$\sin \alpha = \frac{\textit{opposite}}{\textit{hypotenuse}} = \frac{a}{c}$$

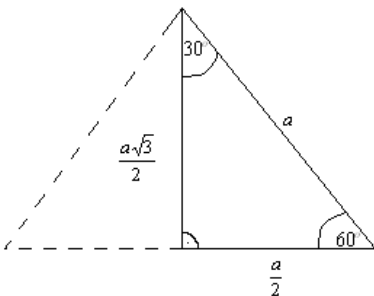
$$\cos \alpha = \frac{\textit{adjacent}}{\textit{hypotenuse}} = \frac{b}{c}$$

$$\textit{tg} \alpha = \frac{\textit{opposite}}{\textit{adjacent}} = \frac{a}{b}$$

$$\textit{ctg} \alpha = \frac{\textit{adjacent}}{\textit{opposite}} = \frac{b}{a}$$

**Take heed:** The symbol sin (cos, tg, ctg) will not indicate any size! Always must have angle!

How to calculate values of trigonometric functions for the angles  $30^\circ$ ,  $45^\circ$  and  $60^\circ$  ?

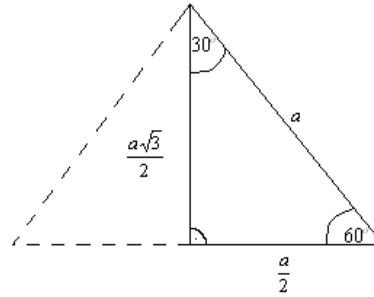


$$\sin 30^\circ = \frac{\textit{opposite}}{\textit{hypotenuse}} = \frac{\frac{a}{2}}{a} = \frac{a}{2a} = \frac{1}{2}$$

$$\cos 30^\circ = \frac{\textit{adjacent}}{\textit{hypotenuse}} = \frac{\frac{a\sqrt{3}}{2}}{a} = \frac{\sqrt{3}}{2}$$

$$\textit{tg} 30^\circ = \frac{\textit{opposite}}{\textit{adjacent}} = \frac{\frac{a}{2}}{\frac{a\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\textit{ctg} 30^\circ = \frac{\textit{adjacent}}{\textit{opposite}} = \frac{\frac{a\sqrt{3}}{2}}{\frac{a}{2}} = \sqrt{3}$$



Now we'll do (by definition) for angle  $60^\circ$  :

$$\sin 60^\circ = \frac{a\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

$$\cos 60^\circ = \frac{a}{2} = \frac{1}{2}$$

$$\textit{tg} 60^\circ = \frac{\sqrt{3}}{\frac{a}{2}} = \frac{\sqrt{3}}{\frac{1}{2}}$$

$$\textit{ctg} 60^\circ = \frac{\frac{a}{2}}{a\sqrt{3}} = \frac{\sqrt{3}}{3}$$

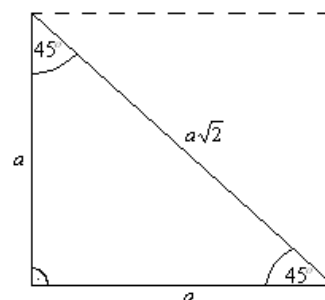
For the value of the trigonometric functions of angle  $45^\circ$  , we will use half of the square.

$$\sin 45^\circ = \frac{\textit{opposite}}{\textit{hypotenuse}} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos 45^\circ = \frac{\textit{adjacent}}{\textit{hypotenuse}} = \frac{a}{a\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\textit{tg} 45^\circ = \frac{\textit{opposite}}{\textit{adjacent}} = \frac{a}{a} = 1$$

$$\textit{ctg} 45^\circ = \frac{\textit{adjacent}}{\textit{opposite}} = \frac{a}{a} = 1$$



In this way we get table:

	$30^\circ$	$45^\circ$	$60^\circ$
$\sin\alpha$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos\alpha$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\operatorname{tg}\alpha$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$
$\operatorname{ctg}\alpha$	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$

Of course, we will table later expand to all corners from  $0^\circ \rightarrow 360^\circ$ .

### Basic trigonometric identity:

$$1) \sin^2 \alpha + \cos^2 \alpha = 1$$

$$2) \operatorname{tg}\alpha = \frac{\sin \alpha}{\cos \alpha}$$

$$3) \operatorname{ctg}\alpha = \frac{\cos \alpha}{\sin \alpha}$$

$$4) \operatorname{tg}\alpha \cdot \operatorname{ctg}\alpha = 1$$

To try to prove some of the identity:

$$1) \sin^2 \alpha + \cos^2 \alpha = \left( \text{by definitions: } \sin \alpha = \frac{a}{c} \text{ and } \cos \alpha = \frac{b}{c} \right) = \frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{c^2}{c^2} = 1$$

$$2) \frac{\sin \alpha}{\cos \alpha} = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{a \cdot c}{b \cdot c} = \frac{a}{b} = \operatorname{tg}\alpha$$

$$4) \operatorname{tg}\alpha \cdot \operatorname{ctg}\alpha = \left( \text{change in the definition, that is } \operatorname{tg}\alpha = \frac{a}{b} \text{ i } \operatorname{ctg}\alpha = \frac{a}{b} \right) = \frac{a}{b} \cdot \frac{b}{a} = 1$$

From the basic identity can be done other equality:

If you go by:

$\sin^2 \alpha + \cos^2 \alpha = 1 \rightarrow$  divide this with  $\cos^2 \alpha$

$$\operatorname{tg} \alpha = \frac{\sin^2 \alpha}{\cos^2 \alpha} + \frac{\cos^2 \alpha}{\cos^2 \alpha} = \frac{1}{\cos^2 \alpha}$$

$\operatorname{tg}^2 \alpha + 1 = \frac{1}{\cos^2 \alpha} \rightarrow$  Express from here  $\cos^2 \alpha$

$$\cos^2 \alpha = \frac{1}{\operatorname{tg}^2 \alpha + 1}$$

Now this change in:

$$\sin^2 \alpha + \cos^2 \alpha = 1$$

$$\sin^2 \alpha + \frac{1}{\operatorname{tg}^2 \alpha + 1} = 1$$

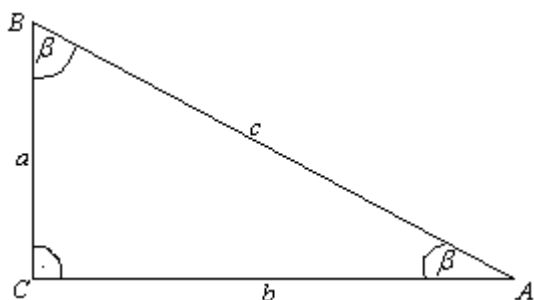
$$\sin^2 \alpha = 1 - \frac{1}{\operatorname{tg}^2 \alpha + 1}$$

$$\sin^2 \alpha = \frac{\operatorname{tg}^2 \alpha + 1 - 1}{\operatorname{tg}^2 \alpha + 1}$$

$$\sin^2 \alpha = \frac{\operatorname{tg}^2 \alpha}{\operatorname{tg}^2 \alpha + 1}$$

These two we will remember and use them in tasks!

Next:



From the image (by definition) is:

$$\sin \alpha = \frac{a}{c}$$

$$\sin \beta = \frac{b}{c}$$

$$\cos \alpha = \frac{b}{c}$$

$$\cos \beta = \frac{a}{c}$$

$$\operatorname{tg} \alpha = \frac{a}{b}$$

$$\operatorname{tg} \beta = \frac{b}{a}$$

$$\operatorname{ctg} \alpha = \frac{b}{a}$$

$$\operatorname{ctg} \beta = \frac{a}{b}$$

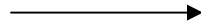
We have new identity:

$$\sin(90^\circ - \alpha) = \cos \alpha$$

$$\cos(90^\circ - \alpha) = \sin \alpha$$

$$\operatorname{tg}(90^\circ - \alpha) = \operatorname{ctg} \alpha$$

$$\operatorname{ctg}(90^\circ - \alpha) = \operatorname{tg} \alpha$$



$$\sin \beta = \cos \alpha$$

$$\cos \beta = \sin \alpha$$

$$\operatorname{tg} \beta = \operatorname{ctg} \alpha$$

$$\operatorname{ctg} \beta = \operatorname{tg} \alpha$$

1) We have cathetus of right angle triangle:  $a=8\text{cm}$  and  $b=6\text{cm}$ . Determine the value of all trigonometric functions for angles  $\alpha$  and  $\beta$ .

Solution:

$$a = 8\text{cm}$$

$$b = 6\text{cm}$$

$$\overline{c^2 = a^2 + b^2}$$

$$c^2 = 8^2 + 6^2$$

$$c^2 = 64 + 36$$

$$c^2 = 100$$

$$c = 10\text{cm}$$

$$\sin \alpha = \frac{a}{c} = \frac{8}{10} = \frac{4}{5} = \cos \beta$$

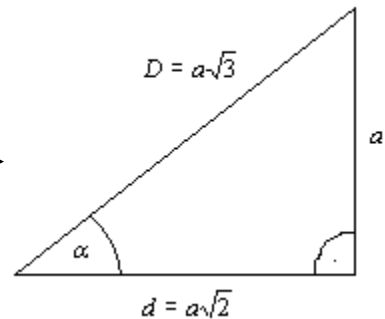
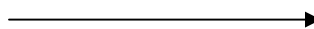
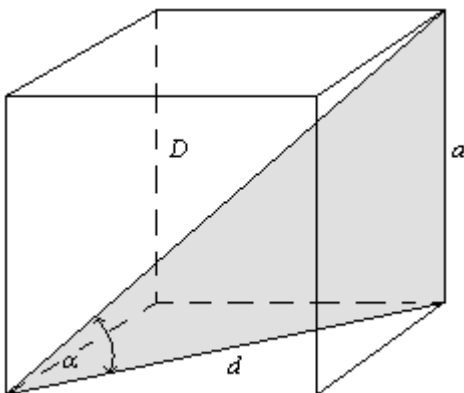
$$\cos \alpha = \frac{b}{c} = \frac{6}{10} = \frac{3}{5} = \sin \beta$$

$$\operatorname{tg} \alpha = \frac{a}{b} = \frac{8}{6} = \frac{4}{3} = \operatorname{ctg} \beta$$

$$\operatorname{ctg} \alpha = \frac{b}{a} = \frac{6}{8} = \frac{3}{4} = \operatorname{tg} \beta$$

2) Calculate the value of trigonometric functions slope angle diagonal of the cube to basis.

Solution:



As we know , the small diagonal is  $d = a\sqrt{2}$  ,a large (body) diagonal is:  $D = a\sqrt{3}$  .

By the definitions:

$$\sin \alpha = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\cos \alpha = \frac{a\sqrt{2}}{a\sqrt{3}} = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}}{3}$$

$$\operatorname{tg} \alpha = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\operatorname{ctg} \alpha = \sqrt{2}$$

3) Calculate unknown, if:  $c = 24\text{cm}$

$$\sin \alpha = 0,8$$

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$$a = ?$$

$$b = ?$$

Solution:

$$\sin \alpha = \frac{a}{c}$$

$$0,8 = \frac{a}{24}$$

$$a = 24 \cdot 0,8$$

$$a = 19,2\text{cm}$$

$$b^2 = c^2 - a^2 \quad \text{Pythagorean theorem}$$

$$b^2 = 24^2 - (19,2)^2$$

$$b^2 = 576 - 368,64$$

$$b^2 = 207,36$$

$$b = 14,4\text{cm}$$

4) Calculate the value of other trigonometric functions, if:

a)  $\sin \alpha = 0,6$

b)  $\cos \alpha = \frac{12}{13}$

c)  $\operatorname{tg} \alpha = 0,225$

**Solution:**

a)  $\sin \alpha = \frac{3}{5}$  because  $0,6 = \frac{6}{10} = \frac{3}{5}$  and we will use  $\sin^2 \alpha + \cos^2 \alpha = 1$

$$\left(\frac{3}{5}\right)^2 + \cos^2 \alpha = 1$$

$$\cos^2 \alpha = 1 - \frac{9}{25}$$

$$\cos^2 \alpha = \frac{16}{25}$$

$$\cos \alpha = \pm \sqrt{\frac{16}{25}}$$

$$\cos \alpha = \pm \frac{4}{5}$$

As the sharp angle is in question:

$$\cos \alpha = +\frac{4}{5}$$

$$\operatorname{tg} \alpha = \frac{\sin \alpha}{\cos \alpha} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$$

$$\operatorname{ctg} \alpha = \frac{1}{\operatorname{tg} \alpha} = \frac{4}{3}$$

b)

$$\cos \alpha = \frac{12}{13}$$

$$\sin^2 \alpha + \cos^2 \alpha = 1$$

$$\sin^2 \alpha + \left(\frac{12}{13}\right)^2 = 1$$

$$\sin^2 \alpha = 1 - \frac{144}{169}$$

$$\sin^2 \alpha = \frac{25}{169}$$

$$\sin \alpha = \pm \sqrt{\frac{25}{169}}$$

$$\sin \alpha = \pm \frac{5}{13}$$

$$\sin \alpha = \frac{5}{13}$$

$$\operatorname{tg} \alpha = \frac{\sin \alpha}{\cos \alpha} = \frac{\frac{5}{13}}{\frac{12}{13}} = \frac{5}{12}$$

$$\operatorname{ctg} \alpha = \frac{12}{5}$$

$$c) \operatorname{tg} \alpha = 0,225 = \frac{225}{1000} = \frac{9}{40}$$

$$\sin^2 \alpha = \frac{\operatorname{tg}^2 \alpha}{\operatorname{tg}^2 \alpha + 1}$$

$$\sin^2 \alpha = \frac{\left(\frac{9}{40}\right)^2}{\left(\frac{9}{40}\right)^2 + 1}$$

$$\sin^2 \alpha = \frac{\frac{81}{1600}}{\frac{81}{1600} + 1}$$

$$\sin^2 \alpha = \frac{\frac{81}{1600}}{\frac{81 + 1600}{1600}}$$

$$\sin^2 \alpha = \frac{81}{1681}$$

$$\sin \alpha = \pm \sqrt{\frac{81}{1681}}$$

$$\sin \alpha = \pm \frac{9}{41}$$

$$\sin \alpha = + \frac{9}{41}$$

$$\cos^2 \alpha = \frac{1}{\operatorname{tg}^2 \alpha + 1}$$

$$\cos^2 \alpha = \frac{1}{\frac{1600}{1681}}$$

$$\cos^2 \alpha = \frac{1600}{1681}$$

$$\cos \alpha = \pm \sqrt{\frac{1600}{1681}}$$

$$\cos \alpha = \pm \frac{40}{41}$$

$$\cos \alpha = + \frac{40}{41}$$

$$\operatorname{ctg} \alpha = \frac{1}{\operatorname{tg} \alpha}$$

$$\operatorname{ctg} \alpha = \frac{40}{9}$$

5) Calculate the value of other trigonometric functions if:

$$\text{a) } \sin \alpha = \frac{a^2 - 9}{a^2 + 9}$$

$$\text{b) } \operatorname{ctg} \alpha = \frac{a^2 - 4}{4a}$$

**Solution: a)**

$$\sin^2 \alpha + \cos^2 \alpha = 1$$

$$\cos^2 \alpha = 1 - \sin^2 \alpha$$

$$\cos^2 \alpha = 1 - \left( \frac{a^2 - 9}{a^2 + 9} \right)^2$$

$$\cos^2 \alpha = 1 - \frac{(a^2 - 9)^2}{(a^2 + 9)^2}$$

$$\cos^2 \alpha = \frac{(a^2 + 9)^2 - (a^2 - 9)^2}{(a^2 + 9)^2}$$

$$\cos^2 \alpha = \frac{a^4 + 18a^2 + 81 - a^4 + 18a^2 - 81}{(a^2 + 9)^2}$$

$$\cos^2 \alpha = \frac{36a^2}{(a^2 + 9)^2}$$

$$\cos \alpha = \sqrt{\frac{36a^2}{(a^2 + 9)^2}}$$

$$\cos \alpha = \frac{6a}{a^2 + 9}$$

$$\operatorname{tg} \alpha = \frac{\sin \alpha}{\cos \alpha}$$

$$\operatorname{tg} \alpha = \frac{a^2 - 9}{a^2 + 9}$$

$$\operatorname{tg} \alpha = \frac{a^2 - 9}{6a}$$

$$\operatorname{tg} \alpha = \frac{a^2 - 9}{6a}$$

$$\operatorname{ctg} \alpha = \frac{6a}{a^2 - 9}$$

**b)**

$$\operatorname{ctg} \alpha = \frac{a^2 - 4}{4a} \Rightarrow \operatorname{tg} \alpha = \frac{4a}{a^2 - 4}$$

$$\sin^2 \alpha = \frac{\operatorname{tg}^2 \alpha}{\operatorname{tg}^2 \alpha + 1}$$

$$\sin^2 \alpha = \frac{\left( \frac{4a}{a^2 - 4} \right)^2}{\left( \frac{4a}{a^2 - 4} \right)^2 + 1}$$

$$\sin^2 \alpha = \frac{16a^2}{\frac{(a^2 - 4)^2}{16a^2} + 1}$$

$$\sin^2 \alpha = \frac{16a^2}{16a^2 + a^4 - 8a^2 + 16}$$

$$\sin^2 \alpha = \frac{16a^2}{a^4 + 8a^2 + 16}$$

$$\sin \alpha = \sqrt{\frac{16a^2}{(a^2 + 4)^2}}$$

$$\sin \alpha = \frac{4a}{a^2 + 4}$$

$$\cos^2 \alpha = \frac{1}{\operatorname{tg}^2 \alpha + 1}$$

$$\cos^2 \alpha = \frac{1}{\left( \frac{4a}{a^2 - 4} \right)^2 + 1}$$

$$\cos^2 \alpha = \frac{1}{\frac{16a^2 + (a^2 - 4)^2}{(a^2 - 4)^2}}$$

$$\cos^2 \alpha = \frac{1}{\frac{(a^2 + 4)^2}{(a^2 - 4)^2}}$$

$$\cos^2 \alpha = \frac{(a^2 - 4)^2}{(a^2 + 4)^2}$$

$$\cos \alpha = \sqrt{\frac{(a^2 - 4)^2}{(a^2 + 4)^2}}$$

$$\cos \alpha = \frac{a^2 - 4}{a^2 + 4}$$

6) Prove that:  $\left(1 + \operatorname{tg}x + \frac{1}{\cos x}\right) \cdot \left(1 + \operatorname{tg}x - \frac{1}{\cos x}\right) = 2\operatorname{tg}x$

**Solution:**

$$\begin{aligned} & \left(1 + \operatorname{tg}x + \frac{1}{\cos x}\right) \cdot \left(1 + \operatorname{tg}x - \frac{1}{\cos x}\right) = \\ & \left(1 + \frac{\sin x}{\cos x} + \frac{1}{\cos x}\right) \cdot \left(1 + \frac{\sin x}{\cos x} - \frac{1}{\cos x}\right) = \\ & \frac{\cos x + \sin x + 1}{\cos x} \cdot \frac{\cos x + \sin x - 1}{\cos x} = \\ & \frac{(\cos x + \sin x)^2 - 1^2}{\cos^2 x} = (1 \text{ will be replaced with } \sin^2 x + \cos^2 x) \\ & \frac{\cos^2 x + 2\cos x \sin x + \sin^2 x - \sin^2 x - \cos^2 x}{\cos^2 x} = \frac{2\cos x \sin x}{\cos^2 x} = \\ & = 2 \frac{\sin x}{\cos x} = 2\operatorname{tg}x \end{aligned}$$

7) Prove that : a)  $\cos^2 18^\circ + \cos^2 36^\circ + \cos^2 54^\circ + \cos^2 72^\circ = 2$

b)  $\operatorname{tg}1^\circ \cdot \operatorname{tg}2^\circ \cdot \operatorname{tg}3^\circ \dots \operatorname{tg}44^\circ \cdot \operatorname{tg}45^\circ \cdot \operatorname{tg}46^\circ \dots \operatorname{tg}89^\circ = 1$

**Proof:**

a)  $\cos^2 18^\circ + \cos^2 36^\circ + \cos^2 54^\circ + \cos^2 72^\circ = 2$

Because  $\alpha + \beta = 90^\circ$ ,  $\cos \alpha = \sin \beta$ ,  $\cos 54^\circ$  replace with  $\sin 36^\circ$ , and  $\cos 72^\circ$  replace with  $\sin 18^\circ$ .  
Then:

$$\cos^2 18^\circ + \cos^2 36^\circ + \cos^2 54^\circ + \cos^2 72^\circ =$$

$$= 1 + 1 = 2$$


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b)

$$\begin{aligned} & \operatorname{tg}1^\circ \cdot \operatorname{tg}2^\circ \cdot \operatorname{tg}3^\circ \dots \operatorname{tg}44^\circ \cdot \operatorname{tg}45^\circ \cdot \operatorname{tg}46^\circ \dots \operatorname{tg}89^\circ = \\ & = \text{Kako je } \operatorname{tg} \alpha = \operatorname{ctg} \beta \text{ } (\alpha + \beta = 90^\circ) \text{ Biće=} \\ & \operatorname{tg}1^\circ \cdot \operatorname{tg}2^\circ \cdot \operatorname{tg}3^\circ \dots \operatorname{tg}44^\circ \cdot \operatorname{tg}45^\circ \cdot \operatorname{ctg}44^\circ \dots \operatorname{ctg}2^\circ \cdot \operatorname{ctg}1^\circ \end{aligned}$$

= **As is:**  $\operatorname{tg} \alpha \cdot \operatorname{ctg} \alpha = 1$

=  $1 \cdot 1 \cdot \dots \cdot \operatorname{tg}45^\circ = 1$

8) Prove that :  $\frac{3}{1 - \sin^6 \alpha - \cos^6 \alpha} = (\operatorname{tg} \alpha + \operatorname{ctg} \alpha)^2$

**Proof:**

$$\frac{3}{1 - \sin^6 x - \cos^6 x} = \frac{3}{1 - (\sin^6 x + \cos^6 x)} =$$

We will attempt to transform expression  $\sin^6 x - \cos^6 x \dots$

$$\sin^2 x - \cos^2 x = 1$$

$$\sin^2 x + \cos^2 x = 1 \quad \leftarrow (A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$$

$$\sin^6 x + 3\sin^4 x \cos^2 x + 3\sin^2 x \cos^4 x + \cos^6 x = 1$$

$$\sin^6 x + 3\sin^2 x \cos^2 x \underbrace{(\sin^2 x + \cos^2 x)}_1 + \cos^6 x = 1$$

So:  $\sin^6 x + \cos^6 x = 1 - 3\sin^2 x \cos^2 x$

Let's go back to the task:

$$\frac{3}{1 - \sin^6 x - \cos^6 x} = \frac{3}{1 - (\sin^6 x + \cos^6 x)} = \frac{3}{1 - 1 + 3\sin^2 x \cos^2 x} = \frac{3}{3\sin^2 x \cos^2 x} = \frac{1}{\sin^2 x \cos^2 x}$$

To see now right side of equation:

$$\begin{aligned} (\operatorname{tg} \alpha + \operatorname{ctg} \alpha)^2 &= \operatorname{tg}^2 \alpha + 2\operatorname{tg} \alpha \operatorname{ctg} \alpha + \operatorname{ctg}^2 \alpha \\ &= \frac{\sin^2 \alpha}{\cos^2 \alpha} + 2 + \frac{\cos^2 \alpha}{\sin^2 \alpha} \\ &= \frac{\sin^4 \alpha + 2\sin^2 \alpha \cos^2 \alpha + \cos^4 \alpha}{\sin^2 \alpha \cos^2 \alpha} \\ &= \frac{(\sin^2 \alpha + \cos^2 \alpha)^2}{\sin^2 \alpha \cos^2 \alpha} \\ &= \frac{1}{\sin^2 \alpha \cos^2 \alpha} \end{aligned}$$

This we have proven that the left and right side are equal.

Condition is:

$$\begin{aligned} 1 - \sin^6 \alpha - \cos^6 \alpha &\neq 0 \\ \sin^6 \alpha - \cos^6 \alpha &\neq 1 \\ 1 - 3\sin^2 \alpha \cos^2 \alpha &\neq 1 \\ \sin^2 \alpha \cos^2 \alpha &\neq 0 \\ \sin \alpha \neq 0 \wedge \cos \alpha &\neq 0 \end{aligned}$$

