

Trigonometric functions of half-angle

Formulas are:

$$1. \sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}} \quad \text{or} \quad 2 \sin^2 \frac{\alpha}{2} = 1 - \cos \alpha$$

$$2. \cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}} \quad \text{or} \quad 2 \cos^2 \frac{\alpha}{2} = 1 + \cos \alpha$$

$$3. \operatorname{tg} \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}}$$

$$4. \operatorname{ctg} \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{1 - \cos \alpha}}$$

Examples:

1) Find $\cos \frac{\alpha}{2}$, if $\sin \alpha = \frac{4}{5}$ and $\alpha \in \left(-\frac{3\pi}{2}, -\pi\right)$.

Solution:

Because: $\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}$, we must find $\cos \alpha$.

$$\sin^2 \alpha + \cos^2 \alpha = 1$$

$$\cos^2 \alpha = 1 - \sin^2 \alpha$$

$$\cos^2 \alpha = 1 - \left(\frac{4}{5}\right)^2$$

$$\cos^2 \alpha = 1 - \frac{16}{25}$$

$$\cos^2 \alpha = \frac{9}{25}$$

$$\cos^2 \alpha = \pm \sqrt{\frac{9}{25}}$$

$$\cos \alpha = \pm \frac{3}{5}$$

We have that $\alpha \in \left(-\frac{3\pi}{2}, \pi\right)$

$$\cos \frac{\alpha}{2} = -\sqrt{\frac{1 - \frac{3}{5}}{2}}$$

$$\cos \frac{\alpha}{2} = -\sqrt{\frac{\frac{2}{5}}{2}}$$

$$\cos \frac{\alpha}{2} = -\frac{1}{\sqrt{5}}$$

$$\cos \frac{\alpha}{2} = -\frac{\sqrt{5}}{5}$$

$$\alpha \in \left(-\frac{3\pi}{2}, -\pi\right) \Rightarrow \frac{\alpha}{2} \in \pi \left(-\frac{3\pi}{4}, \frac{\pi}{2}\right)$$

This tells us that angle is from the II quarter.

$$\cos \alpha = -\frac{3}{5}$$

2) Find $\cos \frac{\alpha}{2}$, if $\sin \alpha = \frac{4\sqrt{2}}{9}$ and $\alpha \in \left(\pi, \frac{3\pi}{2}\right)$.

Solution:

First, we must find $\cos \alpha$

$$\cos^2 \alpha = 1 - \sin^2 \alpha$$

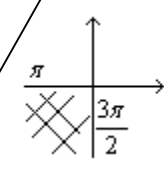
$$\cos^2 \alpha = 1 - \left(-\frac{4\sqrt{2}}{9}\right)^2$$

$$\cos^2 \alpha = 1 - \frac{16 \cdot 2}{81}$$

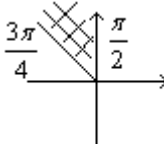
$$\cos^2 \alpha = 1 - \frac{32}{81}$$

$$\cos^2 \alpha = \frac{49}{81}$$

$$\cos \alpha = \pm \frac{7}{9}$$

$$\alpha \in \left(\pi, \frac{3\pi}{2}\right) \Rightarrow$$


$$\cos \alpha = -\frac{7}{9}$$

$$\alpha \in \left(\pi, \frac{3\pi}{2}\right) \Rightarrow \frac{\alpha}{2} \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$$


For $\sin \frac{\alpha}{2}$ we must take +

$$\sin \frac{\alpha}{2} = +\sqrt{\frac{1 - \cos \alpha}{2}} = +\sqrt{\frac{1 + \frac{7}{9}}{2}}$$

$$\sin \frac{\alpha}{2} = +\frac{4}{3\sqrt{2}}$$

$$\sin \frac{\alpha}{2} = \frac{4}{3\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{4\sqrt{2}}{3 \cdot 2}$$

$$\sin \frac{\alpha}{2} = \frac{2\sqrt{2}}{3}$$

3) Prove that:

a) $\operatorname{tg} \frac{\alpha}{2} = \frac{1 - \cos \alpha}{\sin \alpha}$ for $\alpha \neq \pi(2k+1), k \in \mathbb{Z}$

b) $\operatorname{tg} \frac{\alpha}{2} = \frac{\sin \alpha}{1 + \cos \alpha}$ for $\alpha \neq \pi(2k+1), k \in \mathbb{Z}$

Solution:

a) In proof we will use: $\left[\sin \alpha = 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \right]$

$$\frac{1 - \cos \alpha}{\sin \alpha} = \frac{2 \sin^2 \frac{\alpha}{2}}{2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}} = \frac{\sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2}} = \operatorname{tg} \frac{\alpha}{2}$$

b)
$$\frac{\sin \alpha}{1 + \cos \alpha} = \frac{2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}}{2 \cos^2 \frac{\alpha}{2}} = \frac{\sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2}} = \operatorname{tg} \frac{\alpha}{2}$$

4) Calculated the value of expression $A = \frac{\sin x + 2 \cos x}{\operatorname{tg} x - \operatorname{ctg} x}$, if $\operatorname{tg} \frac{x}{2} = 2$

Solution:

First, we will use $\operatorname{tg} \frac{x}{2} = 2$, and find $\cos x$

$$\operatorname{tg} \frac{x}{2} = \sqrt{\frac{1 - \cos x}{1 + \cos x}}$$

$$\sqrt{\frac{1 - \cos x}{1 + \cos x}} = 2$$

$$\frac{1 - \cos x}{1 + \cos x} = 4$$

$$1 - \cos x = 4(1 + \cos x)$$

$$1 - \cos x = 4 + 4 \cos x$$

$$-\cos x - 4 \cos x = 4 - 1$$

$$-5 \cos x = 3$$

$$\cos x = -\frac{3}{5}$$

$$\sin^2 x = 1 - \cos^2 x$$

$$\sin^2 x = 1 - \frac{9}{25}$$

$$\sin^2 x = \frac{16}{25}$$

$$\sin x = \pm \frac{4}{5}$$

Angle is from the II quarter.

$$\sin x = +\frac{4}{5}$$

$$\operatorname{tg} x = \frac{\sin x}{\cos x} = \frac{\frac{4}{5}}{-\frac{3}{5}} = -\frac{4}{3}$$

$$\operatorname{ctg} x = -\frac{3}{4}$$

$$A = ?$$

$$A = \frac{\frac{4}{5} + 2 \cdot \left(-\frac{3}{5}\right)}{-\frac{4}{3} + \frac{3}{4}}$$

$$A = \frac{-\frac{2}{5}}{-\frac{7}{12}}$$

$$A = \frac{24}{35}$$

5) **Prove:**

a) $\frac{1 + \sin x - \cos x}{1 + \sin x + \cos x} = \operatorname{tg} \frac{x}{2}$

b) $\frac{2 \sin x - \sin 2x}{2 \sin x + \sin 2x} = \operatorname{tg}^2 \frac{x}{2}$

Solution:

a)

$$\frac{1 + \sin x - \cos x}{1 + \sin x + \cos x} = \text{The idea is : } \begin{cases} 1 - \cos x = 2 \sin^2 \frac{x}{2} \\ 1 + \cos x = 2 \cos^2 \frac{x}{2} \\ \sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} \end{cases}$$

$$\begin{aligned} &= \frac{2 \sin^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \cos^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}} = \\ &= \frac{2 \sin \frac{x}{2} \left(\sin \frac{x}{2} + \cos \frac{x}{2} \right)}{2 \cos \frac{x}{2} \left(\cos \frac{x}{2} + \sin \frac{x}{2} \right)} \\ &= \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} = \operatorname{tg} \frac{x}{2} \end{aligned}$$

b)

$$\begin{aligned} \frac{2 \sin x - \sin 2x}{2 \sin x + \sin 2x} &= \frac{2 \sin x - 2 \sin x \cos x}{2 \sin x + 2 \sin x \cos x} = \\ &= \frac{2 \sin x (1 - \cos x)}{2 \sin x (1 + \cos x)} = \\ &= \frac{2 \sin^2 \frac{x}{2}}{2 \cos^2 \frac{x}{2}} = \frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} = \operatorname{tg}^2 \frac{x}{2} \end{aligned}$$

6) Figure without the use of mathematical tools : $\operatorname{tg} 7^\circ 30' = ?$

Solution:

The idea is to use just proven equality: $\operatorname{tg} \frac{\alpha}{2} = \frac{\sin \alpha}{1 + \cos \alpha}$

$$\text{So: } \operatorname{tg} 7^\circ 30' = \operatorname{tg} \frac{15^\circ}{2} = \frac{\sin 15^\circ}{1 + \cos 15^\circ}$$

Now , we have to find $\sin 15^\circ$ and $\cos 15^\circ$

$$\sin 15^\circ = \sin \frac{30^\circ}{2} = \sqrt{\frac{1 - \cos 30^\circ}{2}} = \sqrt{\frac{1 - \frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{2 - \sqrt{3}}{4}} \longrightarrow \sin 15^\circ = \frac{\sqrt{2 - \sqrt{3}}}{2}$$

This is true, but it's a little complicated.....

This can be done in more convenient way:

$$\begin{aligned} \sin 15^\circ &= \sin(45^\circ - 30^\circ) = \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ = \\ &= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{2}(\sqrt{3} - 1)}{4} \end{aligned}$$

$$\begin{aligned} \cos 15^\circ &= \cos(45^\circ - 30^\circ) = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ = \\ &= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{2}(\sqrt{3} + 1)}{4} \end{aligned}$$

$$\begin{aligned} \operatorname{tg} 7^\circ 30' &= \frac{\frac{\sqrt{2}(\sqrt{3} - 1)}{4}}{1 + \frac{\sqrt{2}(\sqrt{3} + 1)}{4}} \\ \operatorname{tg} 7^\circ 30' &= \frac{\sqrt{6} - \sqrt{2}}{\sqrt{6} + \sqrt{2} + 5} = \text{double rationalization (see in the root)} \end{aligned}$$

$$\operatorname{tg} 7^\circ 30' = \sqrt{6} - \sqrt{3} + \sqrt{2} - 2$$

7) If $\operatorname{tg} \frac{x}{2} = t$, calculated $\sin x$, $\cos x$ and $\operatorname{tg} x$ using t .

Solution:

This will be a replasement in trigonometric integrals, so pay attention!

$$\sin x = \frac{\sin x}{1} = \frac{2 \sin \frac{x}{2} \cos \frac{x}{2}}{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}} = \frac{\cos^2 \frac{x}{2} \left(2 \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} \right)}{\cos^2 \frac{x}{2} \left(\frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} + 1 \right)}$$

$$= \frac{2 \operatorname{tg} \frac{x}{2}}{\operatorname{tg}^2 \frac{x}{2} + 1} = \frac{2t}{t^2 + 1} = \frac{2t}{1 + t^2}$$

$$\cos x = \frac{\cos x}{1} = \frac{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}} = \frac{\cos^2 \frac{x}{2} \left(1 - \frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} \right)}{\cos^2 \frac{x}{2} \left(\frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} + 1 \right)}$$

$$= \frac{1 - \operatorname{tg}^2 \frac{x}{2}}{1 + \operatorname{tg}^2 \frac{x}{2}} = \frac{1 - t^2}{1 + t^2}$$

$$\operatorname{tg} x = \frac{\sin x}{\cos x} = \frac{\frac{2t}{1+t^2}}{\frac{1-t^2}{1+t^2}} = \frac{2t}{1-t^2}$$

8) Calculate $\frac{\sin 160^\circ}{\cos^4 40^\circ - \sin^4 40^\circ} = ?$

Solution:

$$\frac{\sin 160^\circ}{\cos^4 40^\circ - \sin^4 40^\circ} = \frac{\sin 160^\circ}{\underbrace{(\cos^2 40^\circ + \sin^2 40^\circ)}_1 (\cos^2 40^\circ - \sin^2 40^\circ)} =$$

$$\frac{\sin 160^\circ}{\cos^2 40^\circ - \sin^2 40^\circ} = \{ \text{this is down formula } \cos^2 - \sin^2 x = \cos 2x \}$$

$$\frac{\sin 160^\circ}{\cos 80^\circ} = \frac{2 \sin 80^\circ \cos 80^\circ}{\cos 80^\circ} = 2 \sin 80^\circ$$

