

Trigonometric functions double angle

Formulas are:

1. $\sin 2\alpha = 2 \sin \alpha \cos \alpha$
2. $\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$
3. $\operatorname{tg} 2\alpha = \frac{2\operatorname{tg} \alpha}{1 - \operatorname{tg}^2 \alpha}$
4. $\operatorname{ctg} 2\alpha = \frac{\operatorname{ctg}^2 \alpha - 1}{2\operatorname{ctg} \alpha}$

Examples:

- 1) **Prove :**
- a) $\sin 2\alpha = \frac{2\operatorname{tg} \alpha}{1 + \operatorname{tg}^2 \alpha}$
 - b) $\cos 2\alpha = \frac{1 - \operatorname{tg}^2 \alpha}{1 + \operatorname{tg}^2 \alpha}$
 - c) $\sin 3\alpha = 3\sin \alpha - 4\sin^3 \alpha$

Proofs:

a) $\sin 2\alpha = \frac{2\operatorname{tg} \alpha}{1 + \operatorname{tg}^2 \alpha}$

$$\sin 2\alpha = \sin \alpha \cos \alpha =$$

$$\sin 2\alpha = \frac{\sin \alpha \cos \alpha}{1} = \frac{2 \sin \alpha \cos \alpha}{\sin^2 \alpha + \cos^2 \alpha} = (\text{common and up and down } \cos^2 \alpha) =$$

$$\frac{\cos^2 \alpha \cdot \frac{2 \sin \alpha}{\cos \alpha}}{\cos^2 \alpha \cdot \left(\frac{\sin^2 \alpha}{\cos^2 \alpha} + 1 \right)} = \frac{2 \operatorname{tg} \alpha}{\operatorname{tg}^2 \alpha + 1} = \frac{2 \operatorname{tg} \alpha}{1 + \operatorname{tg}^2 \alpha}$$

b)

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = \frac{\cos^2 \alpha - \sin^2 \alpha}{1} = \frac{\cos^2 \alpha - \sin^2 \alpha}{\sin^2 \alpha + \cos^2 \alpha} =$$

$$= \frac{\cos^2 \alpha \left(1 - \frac{\sin^2 \alpha}{\cos^2 \alpha} \right)}{\cos^2 \alpha \left(\frac{\sin^2 \alpha}{\cos^2 \alpha} + 1 \right)} = \frac{1 - \operatorname{tg}^2 \alpha}{\operatorname{tg}^2 \alpha + 1} = \frac{1 - \operatorname{tg}^2 \alpha}{1 + \operatorname{tg}^2 \alpha},$$

$$\text{c) } \sin 3\alpha = 3\sin \alpha - 4\sin^3 \alpha$$

$$\begin{aligned} \sin 3\alpha &= \sin(2\alpha + \alpha) \rightarrow \text{Use the formula } \sin(2\alpha + \alpha) \\ &= \sin 2\alpha \cos \alpha + \cos 2\alpha \sin \alpha \rightarrow \text{Now the formula for double angle} \\ &= (2\sin \alpha \cos \alpha) \cos \alpha + (\cos^2 \alpha - \sin^2 \alpha) \cdot \sin \alpha \\ &= 2\sin \alpha \cos^2 \alpha + \sin \alpha \cos^2 \alpha - \sin^3 \alpha \\ &= 3\sin \alpha \cos^2 \alpha - \sin^3 \alpha \\ &(\text{from } \sin^2 \alpha + \cos^2 \alpha = 1 \text{ is } \cos^2 \alpha = 1 - \sin^2 \alpha) \\ &= 3\sin \alpha(1 - \sin^2) - \sin^3 \alpha \\ &= 3\sin \alpha - 3\sin^3 \alpha - \sin^3 \alpha \\ &= 3\sin \alpha - 4\sin^3 \alpha \end{aligned}$$

2) We have: $\cos \alpha = \frac{4}{5}$. Find the value of double angles, if α is in the fourth quarter.

Solution:

First, we calculate $\sin \alpha$:

$$\sin^2 \alpha + \cos^2 \alpha = 1$$

$$\sin^2 \alpha = 1 - \cos^2 \alpha$$

$$\sin^2 \alpha = 1 - \left(\frac{4}{5}\right)^2$$

$$\sin^2 \alpha = 1 - \frac{16}{25}$$

$$\sin^2 \alpha = \frac{9}{25}$$

$$\sin \alpha = \pm \sqrt{\frac{9}{25}}$$

$$\sin \alpha = \pm \frac{3}{5} \longrightarrow \text{As the angle } \alpha \text{ is from IV quarter. We'll take that } \sin \alpha = -\frac{3}{5}$$

$$\sin 2\alpha = 2\sin \alpha \cos \alpha$$

$$= 2\left(-\frac{3}{5}\right) \cdot \frac{4}{5}$$

$$= -\frac{24}{25}$$

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$$

$$= \left(\frac{4}{5}\right)^2 - \left(-\frac{3}{5}\right)^2 = \frac{16}{25} - \frac{9}{25} = \frac{7}{25}$$

$$\text{tg } 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha} = \frac{-\frac{24}{25}}{\frac{7}{25}} = -\frac{24}{7}$$

3) We have $\sin \alpha = 0,6$ and α is in I quarter. Find the value of double angles.

Solution:

$$\sin^2 \alpha + \cos^2 \alpha = 1$$

$$\cos^2 \alpha = 1 - \sin^2 \alpha$$

$$\cos^2 \alpha = 1 - (0,6)^2$$

$$\cos^2 \alpha = 1 - 0,36$$

$$\cos^2 \alpha = 0,64$$

$$\cos \alpha = \pm \sqrt{0,64}$$

$$\cos \alpha = \pm 0,8$$

$$\cos \alpha = +0,8$$

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

$$= 2 \cdot 0,6 \cdot 0,8$$

$$= 2 \cdot \frac{3}{5} \cdot \frac{4}{5} = \frac{24}{25}$$

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$$

$$= \left(\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{7}{25}$$

$$\frac{24}{25}$$

$$\operatorname{tg} 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha} = \frac{24}{7}$$

$$\frac{24}{25}$$

$$\operatorname{tg} 2\alpha = \frac{24}{7}$$

4) Prove:

a) $\sin 15^\circ \cos 15^\circ = \frac{1}{4}$

b) $1 - 4 \sin^2 \alpha \cos^2 \alpha = \cos^2 2\alpha$

Solution:

a)

$$\sin 15^\circ \cos 15^\circ = \left\{ \text{add } \frac{2}{2} \right\} = \frac{2 \sin 15^\circ \cos 15^\circ}{2} = \{ \text{formula: } \sin 2\alpha = 2 \sin \alpha \cos \alpha \} = \frac{\sin 30^\circ}{2} = \frac{\frac{1}{2}}{2} = \frac{1}{4}$$

b) $1 - 4 \sin^2 \alpha \cos^2 \alpha = \cos^2 2\alpha$

$$1 - 4 \sin^2 \alpha \cos^2 \alpha = \cos^2 \alpha \{ \text{formula } \sin 2\alpha = 2 \sin \alpha \cos \alpha, \text{ it is: } 4 \sin^2 \alpha \cos^2 \alpha = \sin^2 2\alpha \}$$

$$= 1 - \sin^2 2\alpha = \cos^2 2\alpha$$

5) Prove:

a) $2 \sin^2 \alpha + \cos 2\alpha = 1$

$$2 \sin^2 \alpha + \cos 2\alpha = 2 \sin^2 \alpha + \cos^2 \alpha - \sin^2 \alpha$$

$$= \sin^2 \alpha + \cos^2 \alpha = 1$$

$$\text{b) } \cos^4 \alpha + \sin^4 \alpha = 1 - 0,5 \sin^2 \alpha$$

To prove this ,start from:

$$\sin^2 \alpha + \cos^2 \alpha = 1 / \text{ all the square}$$

$$\sin^4 \alpha + 2 \sin^2 \alpha \cos^2 \alpha + \cos^4 \alpha = 1$$

$$\sin^4 \alpha + \cos^4 \alpha = 1 - \sin^2 \alpha \cos^2 \alpha$$

$$\sin^4 \alpha + \cos^4 \alpha = 1 - \frac{4 \sin^2 \alpha \cos^2 \alpha}{2}$$

$$\sin^4 \alpha + \cos^4 \alpha = 1 - \frac{1}{2} \sin^2 \alpha$$

$$\sin^4 \alpha + \cos^4 \alpha = 1 - 0,5 \sin^2 2\alpha$$

$$\text{6) Prove identity: } \cos 4\alpha + 4 \cos 2\alpha + 3 = 8 \cos^4 \alpha$$

Proof: We will go from left, to prove right side.

$$\cos 4\alpha + 4 \cos 2\alpha + 3 = 8 \cos^4 \alpha$$

$$\cos 4\alpha + 4 \cos 2\alpha + 3 =$$

$$\cos 2 \cdot (2\alpha) + 4(\cos^2 \alpha - \sin^2 \alpha) + 3 =$$

$$\cos^2 (2\alpha) - \sin^2 (2\alpha) + 4 \cos^2 \alpha - 4 \sin^2 \alpha + 3 =$$

$$(\cos^2 \alpha - \sin^2 \alpha)^2 - (2 \sin \alpha \cos \alpha)^2 + 4 \cos^2 \alpha - 4 \sin^2 \alpha =$$

$$(\cos^2 \alpha - (1 - \cos^2 \alpha))^2 - 4 \sin^2 \alpha \cos^2 \alpha + 4 \cos^2 \alpha - 4 \sin^2 \alpha + 3 =$$

$$(2 \cos^2 \alpha - 1)^2 - 4 \cos^2 \alpha (1 - \cos^2 \alpha) + 4 \cos^2 \alpha - 4(1 - \cos^2 \alpha) + 3 =$$

$$4 \cos^4 \alpha - 4 \cos^2 \alpha + 1 - 4 \cos^2 \alpha + 4 \cos^4 \alpha + 4 \cos^2 \alpha - 4 + 4 \cos^2 \alpha + 3 =$$

$$= 8 \cos^4 \alpha$$

$$\text{7) If } \sin \frac{x}{2} + \cos \frac{x}{2} = 1,4 \text{ calculate } \sin \alpha$$

Solution:

$$\sin \frac{x}{2} + \cos \frac{x}{2} = 1,4 / ()^2$$

$$\sin^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2} + \cos^2 \frac{x}{2} = 1,96$$

$$\underbrace{2 \sin \frac{x}{2} \cos \frac{x}{2}}_{\sin \alpha}$$

$$1 + \sin \alpha = 1,96$$

$$\sin \alpha = 1,96 - 1$$

$$\sin \alpha = 0,96$$

8) Introduce $tg3\alpha$ as function of $tg\alpha$

Solution:

$$\begin{aligned}
 tg3\alpha &= tg(2\alpha + \alpha) = \frac{tg2\alpha + tg\alpha}{1 + tg2\alpha \cdot tg\alpha} = \\
 &= \frac{\frac{2tg\alpha}{1 - tg^2\alpha} + tg\alpha}{1 + tg\alpha \cdot \frac{2tg\alpha}{1 - tg^2\alpha}} = \frac{\frac{2tg\alpha + tg\alpha(1 - tg^2\alpha)}{1 - tg^2\alpha}}{\frac{1 - tg^2\alpha + 2tg^2\alpha}{1 - tg^2\alpha}} \\
 &= \frac{2tg\alpha + tg\alpha - tg^3\alpha}{1 + tg^2\alpha} = \frac{3tg\alpha - tg^3\alpha}{1 + tg^2\alpha}
 \end{aligned}$$

9) Prove identity: $\frac{1 + \sin 2\alpha}{\sin \alpha + \cos \alpha} = \sqrt{2} \cos\left(\frac{\pi}{4} - \alpha\right)$

Proof:

$$\begin{aligned}
 \frac{1 + \sin 2\alpha}{\sin \alpha + \cos \alpha} &= \frac{\overbrace{\sin^2 \alpha + \cos^2 \alpha} + 2 \sin \alpha \cos \alpha}{\sin \alpha + \cos \alpha} = \frac{(\sin \alpha + \cos \alpha)^2}{\sin \alpha + \cos \alpha} \\
 &= \sin \alpha + \cos \alpha = (\text{in both addend will add } \frac{2}{2} = \frac{\sqrt{2}^2}{2}) \\
 &\frac{\sqrt{2}^2}{2} \sin \alpha + \frac{\sqrt{2}^2}{2} \cos \alpha = (\text{as a common } \sqrt{2}) \\
 &\sqrt{2} \left(\frac{\sqrt{2}}{2} \sin \alpha + \frac{\sqrt{2}}{2} \cos \alpha \right) = (\text{because of: } \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2} \text{ and } \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}) \\
 &\sqrt{2} \left(\sin \frac{\pi}{4} \sin \alpha + \cos \frac{\pi}{4} \cos \alpha \right) = \\
 &\sqrt{2} \left(\cos \frac{\pi}{4} \cos \alpha + \sin \alpha \sin \frac{\pi}{4} \right) = \{ \text{This is a formula for } \cos(\alpha - \beta) \} \\
 &\sqrt{2} \cos\left(\frac{\pi}{4} - \alpha\right)
 \end{aligned}$$