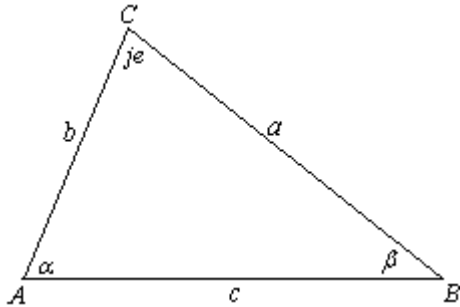


## Sine and cosine theorem

### Sine theorem:(law of sines)



$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma} = 2R$$

For a triangle with length of sides  $a$ ,  $b$ ,  $c$  and angles of  $\alpha$ ,  $\beta$ ,  $\gamma$  respectively, the ratio of the length of side  $a$  to the sine of its corresponding angle  $\alpha$  is equal to the ratio of the length of side  $b$  to the sine of its corresponding angle  $\beta$ .

Sine theorem applies:

- When we have two angle and length of one side
- When we have length of two sides and angle opposed to one of the sides

### Cosine theorem:(law of cosines)

For a triangle with length of sides  $a$ ,  $b$ ,  $c$  and angles of  $\alpha$ ,  $\beta$ ,  $\gamma$  respectively:

$$a^2 = b^2 + c^2 - 2bc \cos \alpha$$

$$b^2 = a^2 + c^2 - 2ac \cos \beta$$

$$c^2 = a^2 + b^2 - 2ab \cos \gamma$$

Cosine theorem applies:

- When we have length of two sides and the angle between them
- When we have length of all three sides

Some important "things" that derive from sinus and cosine theorem are:

**Triangle area:**

$$P = \frac{1}{2} a \sin \alpha$$

$$P = \frac{1}{2} b \sin \beta$$

$$P = \frac{1}{2} c \sin \gamma$$

$P = \frac{a \cdot b \cdot c}{4R}$ , where is  $s = \frac{a+b+c}{2}$ , R is radius of circles described about the triangle

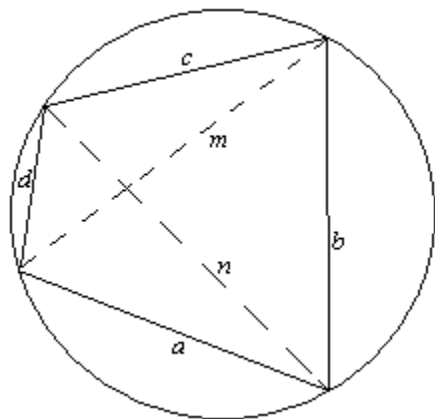
**Median triangle:**

$$t_a = \frac{\sqrt{2b^2 + 2c^2 - a^2}}{2}$$

$$t_b = \frac{\sqrt{2c^2 + 2a^2 - b^2}}{2}$$

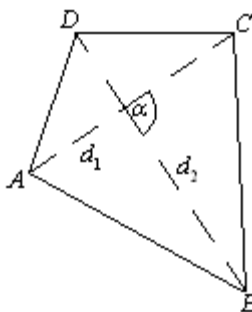
$$t_c = \frac{\sqrt{2a^2 + 2b^2 - c^2}}{2}$$

**Ptolomerev theorem**  $m \cdot n = ac + bd$  **where is:**



**Area of the quadrilateral is:**

$$P = \frac{1}{2} d_1 \cdot d_2 \sin \alpha$$



**EXAMPLES:**

- 1) In the triangle ABC is  $\alpha = 45^\circ$ ,  $\beta = 60^\circ$  and the radius of the circle described  $R = 2\sqrt{6}$ . Determine the other basic elements without the use of table.

**Solution:**

$$\alpha = 45^\circ$$

$$\beta = 60^\circ$$

$$R = 2\sqrt{6}$$

First, we find angle  $\gamma$

$$\alpha + \beta + \gamma = 180^\circ$$
$$\gamma = 180^\circ - (45^\circ + 60^\circ)$$
$$\gamma = 75^\circ$$

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma} = 2R$$

$$\frac{a}{\sin \alpha} = 2R \Rightarrow a = 2R \sin \alpha$$
$$a = 2 \cdot 2\sqrt{6} \sin 45^\circ$$
$$a = 4\sqrt{6} \cdot \frac{\sqrt{2}}{\sqrt{2}} = 2\sqrt{12} = 4\sqrt{3}$$
$$a = 4\sqrt{3}$$

$$\begin{aligned} \frac{b}{\sin \beta} = 2R &\Rightarrow b = 2R \sin \beta \\ b &= 2 \cdot 2\sqrt{6} \sin 60^\circ \\ b &= 4\sqrt{6} \cdot \frac{\sqrt{3}}{2} = 2\sqrt{18} = 6\sqrt{3} \\ b &= 6\sqrt{3} \end{aligned}$$

$$\begin{aligned} \frac{c}{\sin \gamma} = 2R &\Rightarrow c = 2R \sin \gamma \\ c &= 2 \cdot R\sqrt{6} \sin 75^\circ \\ c &= 4\sqrt{6} \cdot \sin(45^\circ + 30^\circ) \\ c &= 4\sqrt{6} \cdot (\sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ) \\ c &= 4\sqrt{6} \cdot \left( \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} \right) \\ c &= 2(3 + \sqrt{3}) \end{aligned}$$

2) Determine length **b** in triangle ABC, if  $a = 2\sqrt{3}$ ,  $c = \sqrt{6}$  and angle  $\beta = 105^\circ$

**Solution:**

$$\begin{array}{l} a = 2\sqrt{3} \text{ cm} \\ c = \sqrt{6} \text{ cm} \\ \beta = 105^\circ \\ \hline b = ? \end{array} \quad \text{Here we use cosine theorem!}$$

$$b^2 = a^2 + c^2 - 2ac \cos \beta$$

Let's go first to find  $\cos 105^\circ = ?$

$$\begin{aligned} \cos 105^\circ &= \cos(60^\circ + 45^\circ) \\ &= \cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ \\ &= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}(1 - \sqrt{3})}{4} \end{aligned}$$

$$b^2 = (2\sqrt{3})^2 + (\sqrt{6})^2 - 2 \cdot 2\sqrt{3} \cdot \sqrt{6} \cdot \frac{\sqrt{2}(1-\sqrt{3})}{4}$$

$$b^2 = 12 + 6 - 6(1-\sqrt{3})$$

$$b^2 = 12 + 6 - 6 + 6\sqrt{3}$$

$$b^2 = 12 + 6\sqrt{3} \longrightarrow 12 + 6\sqrt{3} = (3 + \sqrt{3})^2$$

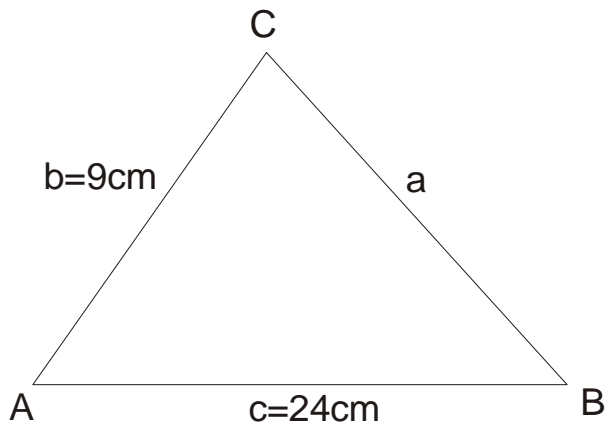
$$b^2 = (3 + \sqrt{3})^2 = 3^2 + 2 \cdot 3\sqrt{3} + \sqrt{3}^2$$

$$b = 3 + \sqrt{3} = 9 + 6\sqrt{3} + 3$$

$$= 12 + 6\sqrt{3}$$

3) In the triangle ABC is  $AB = 24\text{cm}$ ,  $AC = 9\text{cm}$  and angle  $\alpha = 60^\circ$ . Find  $BC = ?$  and  $R = ?$

**Solution:**



$$b = 9\text{cm}$$

$$c = 24\text{cm}$$

$$\alpha = 60^\circ$$

$$a = ?, R = ?$$

$$a^2 = b^2 + c^2 - 2bc \cos \alpha$$

$$a^2 = 9^2 + 24^2 - 2 \cdot 9 \cdot 24 \cdot \cos 60^\circ$$

$$a^2 = 81 + 576 - 2 \cdot 9 \cdot 24 \cdot \frac{1}{2}$$

$$a^2 = 441$$

$$a = \sqrt{441}$$

$$a = 21\text{cm}$$

$$\frac{a}{\sin \alpha} = 2R \Rightarrow \frac{21}{\sin 60^\circ} = 2R$$

$$\frac{21}{\frac{\sqrt{3}}{2}} = 2R$$

$$2R = \frac{42}{\sqrt{3}}$$

$$R = \frac{21}{\sqrt{3}}$$

$$R = \frac{21}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$

$$R = \frac{21\sqrt{3}}{\sqrt{3}}$$

$$R = 7\sqrt{3} \text{ cm}$$

4) In the triangle ABC is:

$$a - b = 3$$

$$\gamma = 60^\circ$$

$$R = \frac{7\sqrt{3}}{3}$$

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$$a, b, c = ?$$

**Solution:**

$$\frac{c}{\sin \gamma} = 2R \Rightarrow c = 2R \sin \gamma$$

$$c = 2 \cdot \frac{7\sqrt{3}}{3} \cdot \sin 60^\circ$$

$$c = 2 \cdot \frac{7\sqrt{3}}{3} \cdot \frac{\sqrt{3}}{2}$$

$$c = 7$$

$$c^2 = a^2 + b^2 - 2ab \cos \gamma$$

$$7^2 = (b+3)^2 + b^2 - 2b(b+3) \cos 60^\circ$$

$$49 = (b+3)^2 + b^2 - 2b(b+3) \cdot \frac{1}{2}$$

$$49 = b^2 + 6b + 9 + b^2 - b^2 - 3b$$

$$b^2 + 3b - 40 = 0 \rightarrow \text{quadratic equation by "b"}$$

$$b_{1,2} = \frac{-3 \pm 13}{2}$$

$$b_1 = 5$$

$b_2 = -8 \rightarrow$  this is not a solution because the length of the page can not be a negative number

So:  $b = 5$

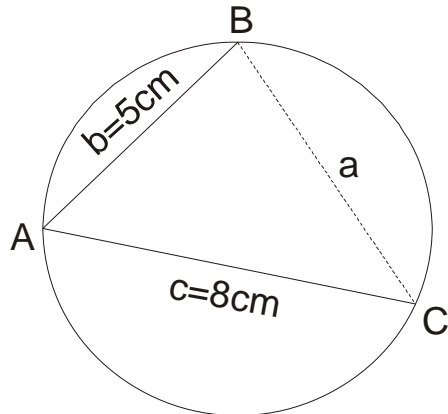
$$a = b + 3$$

$$a = 5 + 3$$

$$a = 8$$

5) In circle we have  $AB=8\text{cm}$  and  $AC=5\text{cm}$ , and angle between them  $\alpha = 60^\circ$ . Find  $R=?$

**Solution:**



$$b = 5\text{cm}$$

$$c = 8\text{cm}$$

$$\alpha = 60^\circ$$

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$$R = ?$$

$$a = b^2 + c^2 - 2bc \cos \alpha$$

$$a^2 = 5^2 + 8^2 - 2 \cdot 5 \cdot 8 \cdot \cos 60^\circ$$

$$a^2 = 25 + 64 - 2 \cdot 40 \cdot \frac{1}{2}$$

$$a^2 = 89 - 40$$

$$a^2 = 49$$

$$a = 7\text{cm}$$

$$\frac{a}{\sin \alpha} = 2R \Rightarrow$$

$$\frac{7}{\sin 60^\circ} = 2R$$

$$2R = \frac{7}{\frac{\sqrt{3}}{2}}$$

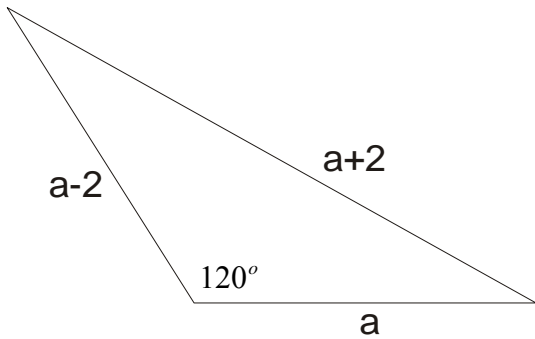
$$R = \frac{7}{\sqrt{3}}$$

$$R = \frac{7}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$

$$R = \frac{7\sqrt{3}}{3}$$

6) If the side lengths of the triangle are  $a-2, a, a+2$ , and angle  $120^\circ$ , find  $a=?$ ,  $b=?$  and  $c=?$

**Solution:**



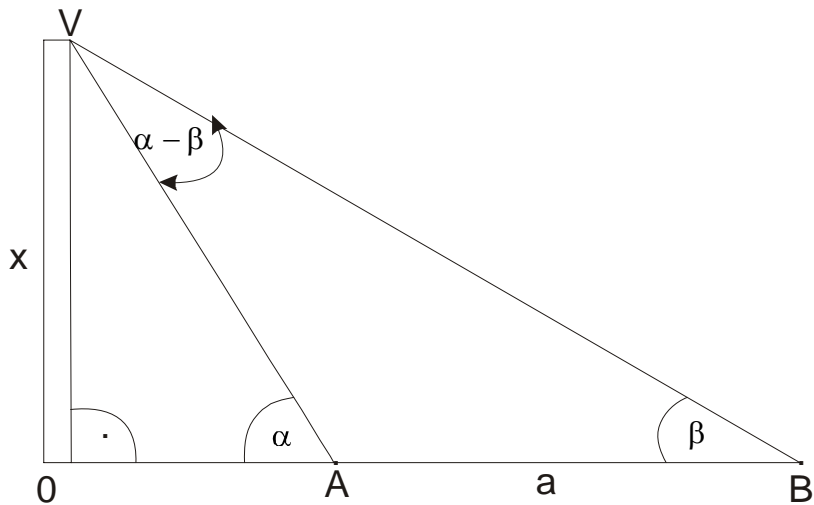
$$\begin{aligned} a &= a \\ b &= a - 2 \\ c &= a + 2 \end{aligned}$$

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab \cos \theta \\ (a+2)^2 &= a^2 + (a-2)^2 - 2a(a-2) \cos 120^\circ \\ (a+2)^2 &= a^2 + (a-2)^2 - 2a(a-2) \cdot \left(-\frac{1}{2}\right) \\ a+2)^2 &= a^2 + (a-2)^2 + a(a-2) \\ a^2 + 4a + 4 &= a^2 + a^2 - 4a + 4 + a^2 - 2a \\ 0 &= 2a^2 - 10a \\ 2a(a-5) &= 0 \\ a = 0 &\rightarrow \text{impossible} \\ a &= 5 \\ b = a - 2 &= 5 - 2 = 3 \\ b &= 3 \\ c = a + 2 & \\ c &= 7 \end{aligned}$$

7) Calculate the amount of factory chimney, which is located on the horizontal soil, if the top of the chimney we can see under angle  $\alpha$  from the point A, from point B under angle  $\beta$ . Points A and B also belong to horizontal plane and their distance is  $AB = a$ .

**Solution:**

**Here is the most important to draft problem!**



With  $OV=X$  mark required height

First, find unknown corners  $\angle OVA$  and  $\angle AVB$

$$\left. \begin{array}{l} \angle OVA = 90^\circ - \alpha \\ \angle OVB = 90^\circ - \beta \end{array} \right\} \Rightarrow \begin{array}{l} AVB = \angle OVB - \angle OVA \\ = (90^\circ - \beta) - (90^\circ - \alpha) \\ = 90^\circ - \beta - 90^\circ + \alpha \\ \angle AVB = \alpha - \beta \end{array}$$

Primenimo sinusnu teoremu na trougao ABV

$$\frac{a}{\sin(\alpha - \beta)} = \frac{AV}{\sin \beta} \Rightarrow AV = \frac{a \sin \beta}{\sin(\alpha - \beta)} \quad \text{next:}$$

$$\begin{aligned} \sin \alpha &= \frac{X}{AV} \Rightarrow X = AV(\sin \alpha) \\ X &= \frac{a \sin \beta \sin \alpha}{\sin(\alpha - \beta)} \\ X &= \frac{a \sin \alpha \sin \beta}{\sin(\alpha - \beta)} \end{aligned}$$

8) In the triangle ABC is  $a - b = 1$ ,  $h_c = \frac{3}{2}$ ,  $R = 4$ . Without the use of table calculate angle  $\alpha$ .

**Solution:**

$$a - b = 1$$

$$h_c = \frac{3}{2}$$

$$R = 4$$

$$\alpha = ?$$

First, we use forms for the area of a triangle:

$$P = \frac{c \cdot h_c}{2}, \quad P = \frac{abc}{4R}$$

$$\text{So:} \quad \frac{c \cdot h_c}{2} = \frac{abc}{4R}$$

$$2ab = 4Rh_c$$

$$ab = 2Rh_c$$

$$ab = 2 \cdot 4 \cdot h_c$$

$$ab = 2 \cdot 4 \cdot \frac{3}{2}$$

$$ab = 12$$

Now a system:

$$a - b = 1$$

$$ab = 12$$

$$a = b + 1$$

$$b(b + 1) = 12$$

$$b^2 + b - 12 = 0$$

$$b_{1,2} = \frac{-1 \pm 7}{2}$$

$$b_1 = 3$$

$$b_2 = -4 \quad \text{impossible}$$

$$\text{So:} \quad b = 3 \Rightarrow a = 3 + 1 = 4 \Rightarrow a = 4$$

$$\frac{a}{\sin \alpha} = 2R \Rightarrow \sin \alpha = \frac{a}{2R}$$

$$\sin \alpha = \frac{4}{8}$$

$$\sin \alpha = \frac{1}{2}$$

$$\alpha = 30^\circ \text{ because } \sin = 30^\circ = \frac{1}{2}$$

$\alpha = 30^\circ$  is solution!

9) Calculate the length of a, b and c triangle if  $P = 3\sqrt{3}$ ,  $\alpha = 60^\circ$ ,  $b+c=7$ .

**Solution:**

$$P = 3\sqrt{3}$$

$$\alpha = 60^\circ$$

$$b+c=7$$

$$a, b, c = ?$$

$$P = \frac{1}{2}bc \sin \alpha$$

$$3\sqrt{3} = \frac{1}{2}bc \sin 60^\circ$$

$$3\sqrt{3} = \frac{1}{2}bc \cdot \frac{\sqrt{3}}{3}$$

$$bc = 12$$

$$b+c=7$$

$$bc=12$$

$$c=7-b \longrightarrow bc=12$$

$$b \cdot (7-b) = 12$$

$$7b - b^2 = 12$$

$$b^2 - 7b + 12 = 0$$

$$b_{1,2} = \frac{7 \pm 1}{2}$$

$$b_1 = 4 \Rightarrow c = 3$$

$$b_2 = 3 \Rightarrow c = 4$$

So, we have two options:

$$b_1 = 4, c = 3 \quad \text{or} \quad b_2 = 3, c = 4$$

Use now cosine theorem:

$$a^2 = b^2 + c^2 - 2bc \cos \alpha$$

$$a^2 = 4^2 + 3^2 - 2 \cdot 4 \cdot 3 \cdot \cos 60^\circ$$

$$a^2 = 16 + 9 - 2 \cdot 12 \cdot \frac{1}{2}$$

$$a^2 = 25 - 12$$

$$a^2 = 13$$

$$a = \sqrt{13}$$