

Systems square equation with two unknown

There are several types of systems:

1) The system of one square and one linear equations with two unknown

From linear equation express x or y (whichever is easier), this change in quadratic equation and solve it...

Example 1. Solve the system: $2x^2 + 2y^2 + 3x - 2 = 0$
 $x - 2y = -2$

Solution:

$$2x^2 + 2y^2 + 3x - 2 = 0$$

$x - 2y = -2$ \longrightarrow from here express x (easier) and change into the square equation

$$\begin{aligned} & \xrightarrow{x = 2y - 2} \\ & 2(2y - 2)^2 + 2y^2 + 6y - 6 - 2 = 0 \\ & 2(4y^2 - 8y + 4) + 2y^2 + 6y - 6 - 2 = 0 \\ & 8y^2 - 16y + 8 + 2y^2 + 6y - 8 = 0 \\ & 10y^2 - 10y = 0 / : 10 \\ & y^2 - y = 0 \\ & y(y - 1) = 0 \\ & y_1 = 0 \quad \vee \quad y - 1 = 0 \\ & \qquad \qquad \qquad y_2 = 1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= 2 \cdot 0 - 2 = -2 \\ x_2 &= 2 \cdot 1 - 2 = 0 \end{aligned}$$

Solutios are: $(x_1, y_1) = (-2, 0)$ and $(x_2, y_2) = (0, 1)$

Example 2. Solve the system: $3x^2 + 2xy + 2y^2 + 3x - 4y = 0$
 $2x - y + 5 = 0$

Solution:

$$3x^2 + 2xy + 2y^2 + 3x - 4y = 0$$

$$2x - y + 5 = 0$$

$y = 2x + 5$

$$3x^2 + 2x(2x + 5) + 2(2x + 5)^2 + 3x - 4(2x + 5) = 0$$
$$3x^2 + 4x^2 + 10x + 2(4x^2 + 20x + 25) + 3x - 8x - 20 = 0$$
$$7x^2 + 10x + 8x^2 + 40x + 50 + 3x - 8x - 20 = 0$$
$$15x^2 + 45x + 30 = 0 / :15$$
$$x^2 + 3x + 2 = 0$$

$a = 1$ $x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm 1}{2}$
 $b = 3$ $x_1 = -1$
 $c = 2$ $x_2 = -2$

Replace x_1 and x_2 in $y = 2x + 5$:

$$y_1 = 2(-1) + 5 = -2 + 5 = 3$$

$$y_2 = 2(-2) + 5 = -4 + 5 = 1$$

Solutions are: $(-1,3)$, $(-2,1)$

2) The system of two square equations, which contain only ax^2 and ay^2 and free members

This is system: $a_1x^2 + b_1y^2 = c_1$
 $a_2x^2 + b_2y^2 = c_2$

The easiest method to solve this system is “the opposite coefficients”.

Example 1. Solve the system: $5x^2 - 6y^2 = 11$
 $7x^2 + 3y^2 = 714$

Solution:

$$5x^2 - 6y^2 = 11$$

$$7x^2 + 3y^2 = 714 \rightarrow \text{multiply with 2}$$

$$\left. \begin{array}{l} 5x^2 - 6y^2 = 11 \\ 14x^2 + 6y^2 = 1428 \end{array} \right\} +$$

$$19x^2 = 1539$$

$$x^2 = 81$$

$$x = \pm\sqrt{81}$$

$$x = \pm 9 \longrightarrow x_1 = +9 \quad \text{i} \quad x_2 = -9$$

$$7x^2 + 3y^2 = 714$$

$$7 \cdot 81 + 3y^2 = 714$$

$$567 + 3y^2 = 714$$

$$3y^2 = 714 - 567$$

$$3y^2 = 147$$

$$y^2 = 49$$

$$y = \pm\sqrt{49}$$

$$y_1 = +7$$

$$y_2 = -7$$

Now, make "combination" : (9,7), (9,-7), (-9,7), (-9,-7) are solutions.

Before becoming acquainted with the new type of system, we have to learn what are **homogeneous** equations.

Its general form is: $Ax^2 + Bxy + Cy^2 = 0$

It can be solved easily with replacement: $x = yz \longrightarrow z = \frac{x}{y}$

$$Ax^2 + Bxy + Cy^2 = 0$$

$$y^2 \left(A \frac{x^2}{y^2} + B \frac{xy}{y^2} + C \frac{y^2}{y^2} \right) = 0$$

$$y^2 \left(A \left(\frac{x}{y} \right)^2 + B \left(\frac{x}{y} \right) + C \right) = 0$$

$$y^2 (Az^2 + Bz + C) = 0$$

$$y = 0 \quad \vee \quad Az^2 + Bz + C = 0$$

$$z_{1,2} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$z_1 = \dots$$

$$z_2 = \dots$$

$$x = z_1 y \quad \text{i} \quad x = z_2 y$$

3) System of two square equations, which is one homogeneous

This system has form:
$$\begin{cases} Ax^2 + Bxy + Cy^2 = 0 \\ ax^2 + bxy + cy^2 + dx + ry + f = 0 \end{cases}$$

Example 1. Solve the system:
$$\begin{aligned} x^2 - 3xy + 2y^2 &= 0 \\ x^2 - 3x - y + 3 &= 0 \end{aligned}$$

Solution:

$$x^2 - 3xy + 2y^2 = 0 \rightarrow \text{homogeneous, first solve it}$$

$$x^2 - 3x - y + 3 = 0$$

$$x^2 - 3xy + 2y^2 = 0$$

$$y^2 \left(\frac{x^2}{y^2} - 3 \frac{x}{y} + 2 \right) = 0 \quad \frac{x}{y} = z \quad \text{replacement} \quad x = zy$$

$$z^2 - 3z + 2$$

$$z_{1,2} = \frac{3 \pm 1}{2}$$

$$z_1 = 2$$

$$z_2 = 1$$

$$\text{For } z_1 = 2 \Rightarrow x = 2y$$

$$x^2 - 3x - y + 3 = 0$$

$$(2y)^2 - 3 \cdot 2y - y + 3 = 0$$

$$4y^2 - 6y - y + 3 = 0$$

$$4y^2 - 7y + 3 = 0$$

$$y_{1,2} = \frac{7 \pm 1}{8}$$

$$y_1 = 1$$

$$y_2 = \frac{6}{8} = \frac{3}{4}$$

$$x_1 = 2 \cdot 1 = 2, \quad x_2 = 2 \cdot \frac{3}{4} = \frac{3}{2}$$

$$(2,1), \quad \left(\frac{3}{2}, \frac{3}{4} \right)$$

$$\text{For } z_1 = 1 \Rightarrow x = y$$

$$x^2 - 3x - y + 3 = 0$$

$$x^2 - 3x - x + 3 = 0$$

$$4y^2 - 4x + 3 = 0$$

$$x_{1,2} = \frac{4 \pm 2}{2}$$

$$x_1 = 3$$

$$x_2 = 1$$

$$y_1 = 3, \quad y_2 = 1$$

$$(3,3), \quad (1,1)$$

Solutions are: $(2,1), \left(\frac{3}{2}, \frac{3}{4} \right), (3,3), (1,1)$

Example 2. Solve the system: $x^2 - xy - 6y^2 = 0$
 $x^2 - 2xy + 2y^2 = 18$

Solution:

$$x^2 - xy - 6y^2 = 0 \rightarrow$$

$$x^2 - 2xy + 2y^2 = 18$$

$$x^2 + xy + 6y^2 = 0$$

$$y^2 \left(\frac{x^2}{y^2} + \frac{x}{y} - 6 \right) = 0$$

$$z^2 + z - 6 = 0 \text{ replacement } \frac{x}{y} = z$$

$$z_{1,2} = \frac{-1 \pm 5}{2}$$

$$z_1 = 2$$

$$z_2 = -3$$

$$x = yz \Rightarrow x = 2y \text{ or } x = -3y$$

$$x = 2y$$

$$x^2 - 2xy + 2y^2 = 18$$

$$(2y)^2 - 2 \cdot 2y \cdot y + 2y^2 = 18$$

$$4y^2 - 4y^2 + 2y^2 = 18$$

$$y^2 = 9$$

$$y = \pm 3$$

$$y_1 = +3 \quad y_2 = -3$$

$$x_1 = 2 \cdot 3 \quad x_2 = 2 \cdot (-3) = -6$$

$$(6, 3) \quad (-6, -3)$$

$$x = -3y$$

$$x^2 - 2xy + 2y^2 = 18$$

$$(-3y)^2 - 2 \cdot (-3y) \cdot y + 2y^2 = 18$$

$$9y^2 + 6y^2 + 2y^2 = 18$$

$$17y^2 = 18$$

$$\rightarrow y^2 = \frac{18}{17}$$

$$y = \pm \sqrt{\frac{18}{17}}$$

$$y_1 = +\sqrt{\frac{18}{17}}$$

$$y_2 = -\sqrt{\frac{18}{17}}$$

$$x_1 = -3\sqrt{\frac{18}{17}}$$

$$x_2 = 3\sqrt{\frac{18}{17}}$$

$$\left(-3\sqrt{\frac{18}{17}}, \sqrt{\frac{18}{17}} \right) \text{ i } \left(3\sqrt{\frac{18}{17}}, -\sqrt{\frac{18}{17}} \right)$$

So, again there are four solutions!

4) Systems that reduce to the homogeneous equations

General form of this system is:

$$\begin{aligned} a_1x^2 + b_2xy + c_1y^2 &= d_1 \\ a_2x^2 + b_2xy + c_2y^2 &= d_2 \end{aligned}$$

The idea is “to destroyed” d_1 and d_2 by opposing coefficients, and to obtain homogeneous equation. Her resolve and create two new systems. **Nothing without examples:**

Example 1. Solve the system:

$$\begin{aligned} 2x^2 - 3xy + 2y^2 &= 4 \\ x^2 + xy + y^2 &= 7 \end{aligned}$$

Solution:

$$\begin{aligned} 2x^2 - 3xy + 2y^2 &= 4 \\ x^2 + xy + y^2 &= 7 \end{aligned} \quad \text{Multiply the first equation with 7, and the other with -4}$$

$$\left. \begin{aligned} 14x^2 - 21xy + 14y^2 &= 28 \\ -4x^2 - 4xy - 4y^2 &= -28 \end{aligned} \right\} +$$

$$10x^2 - 25xy + 10y^2 = 0 / : 5$$

$$2x^2 - 5xy + 2y^2 = 0 \rightarrow \text{We have received a homogeneous equation!}$$

$$2\frac{x^2}{y^2} - 5\frac{x}{y} + 2 = 0 \quad \text{replacement } \frac{x}{y} = z$$

$$2z^2 - 5z + 2 = 0$$

$$z_{1,2} = \frac{5 \pm 3}{4}$$

$$z_1 = 2$$

$$z_2 = \frac{1}{2}$$

$$\frac{x}{y} = 2 \quad \text{or} \quad \frac{x}{y} = \frac{1}{2}$$

$$x = 2y \quad \text{or} \quad y = 2x$$

Now select one of the leading two equations (one with less numbers) and create two new system:

$$x = 2y$$

$$x^2 + xy + y^2 = 7$$

$$(2y)^2 + 2y \cdot y + y^2 = 7$$

$$4y^2 + 2y^2 + y^2 = 7$$

$$7y^2 = 7$$

$$y^2 = 1$$

$$y = \pm 1$$

$$y_1 = 1$$

$$y_2 = -1$$

$$x_1 = 2 \cdot y_1 = 2$$

$$x_2 = 2 \cdot (-1) = -2$$

$$y = 2x$$

$$x^2 + xy + y^2 = 7$$

$$x^2 + x \cdot 2x + (2x)^2 = 7$$

$$x^2 + 2x^2 + 4x^2 = 7$$

$$7x^2 = 7$$

$$x^2 = 1$$

$$x = \pm 1$$

$$x_1 = 1$$

$$x_2 = -1$$

$$y_1 = 2x_1 = 2$$

$$y_2 = 2 \cdot (-1) = -2$$

Finally solutions are: (2,1), (-2,-1), (1,2), (-1,-2)