Irrational equations

The irrational equation include the equation in which the unknown is “in the root”.

In the general case, this equation can not be solved. We will examine some commoner cases.

Important: equation $\sqrt{a(x)} = b(x)$ **is equivalent to system** $a(x) = b^2(x) \wedge b(x) \geq 0$**.**

Example 1. Solve the equation : $\sqrt{x+7} = x+1$

Solution:

$$\sqrt{x+7} = x+1$$

$$x+7 = (x+1)^2 \quad \wedge \quad x+1 \geq 0 \quad \wedge \quad x+7 \geq 0 \longrightarrow \text{this because of the root}$$

$$x+7 = x^2+2x+1 \quad \wedge \quad x \geq -1 \quad \wedge \quad x \geq -7$$

$$x^2+2x+1-x-7=0$$

$$x^2+x-6=0$$

$$a = 1 \qquad x_{1,2} = \frac{-1 \pm 5}{2}$$

$$b = 1 \qquad x_1 = 2$$

$$c = -6 \qquad x_2 = -3$$

We need to verify whether the solutions are "good"! $x \geq -1$ and $x \geq -7$ are conditions.

$x_1 = 2$ is “good” because $2 \geq -1$ and $2 \geq -7$

$x_1 = -3$ is not “good” because $-3 \geq -1$ is not true! So, the only solution is x = 2.

Example 2. Solve the equation : $1+\sqrt{x^2-9} = x$

Solution:

$1+\sqrt{x^2-9} = x$ Leave the root on one side, and without roots switch to the other side!

$$\sqrt{x^2-9} = x-1$$

$\sqrt{a(x)} = b(x)$ **is equivalent to system** $a(x) = b^2(x) \wedge b(x) \geq 0$ **.**

$$x^2 - 9 = (x-1)^2 \quad \wedge \quad x - 1 \geq 0 \quad \wedge \quad x^2 - 9 \geq 0$$

$x^2 - 9 = x^2 - 2x + 1$

$2x = 1 + 9$

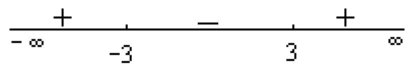
$2x = 10$

$x = 5$

$x \geq 1$

$x_1 = 3$

$x_2 = -3$



$x \in (-\infty, -3] \cup [3, \infty)$

Be sure to check that solution satisfies the conditions: $x \geq 1$ and $x \in (-\infty, -3] \cup [3, \infty)$

x = 5 is a solution , because $5 \geq 1$ and $5 \in [3, \infty)$

Example 3. Solve the equation : $\sqrt{12 - x\sqrt{x^2 - 8}} = 3$

Solution:

$\sqrt{12 - x\sqrt{x^2 - 8}} = 3 /()^2 \rightarrow \underbrace{12 - x\sqrt{x^2 - 8} \geq 0}_{1.condition} \wedge \underbrace{x^2 - 8 \geq 0}_{2.condition}$

$12 - x\sqrt{x^2 - 8} = 9$

$-x\sqrt{x^2 - 8} = 9 - 12$

$-x\sqrt{x^2 - 8} = -3 \rightarrow \sqrt{x^2 - 8} = \dfrac{3}{x} \Rightarrow \dfrac{3}{x} \geq 0 \Rightarrow \underbrace{x > 0}_{3.condition}$

$x - \sqrt{x^2 - 8} = 3 /()^2$

$x^2(x^2 - 8) = 9$

$x^4 - 8x^2 - 9 = 0$

$x^4 - 8x^2 - 9 = 0 \rightarrow$ replacement $x^2 = t$

$t^2 - 8t - 9 = 0$

$t_{1,2} = \dfrac{8 \pm 10}{2}$

$t_1 = 9$

$t_2 = -1$

$x^2 = 9 \vee x^2 = -1$

$x_{3,4} = \pm i$

$x_1 = 3, x_2 = -3$

We need to verify whether the solutions are "good". $x_1 = 3, x_2 = -3$ are solutions.

Replace the solutions in the home equation, to see whether they are"good"!

$$\sqrt{12 - x\sqrt{x^2 - 8}} = 3$$
$$\sqrt{12 - 3\sqrt{3^2 - 8}} = 3$$
$$\sqrt{12 - 3\cdot 1} = 3$$
$$\sqrt{9} = 3$$
$$3 = 3$$

So , x = 3 is the solution

$$x = -3 \Rightarrow \sqrt{12 - x\sqrt{x^2 - 8}} = 3$$
$$\sqrt{12 + 3\sqrt{9 - 8}} = 3$$
$$\sqrt{12 + 3} = 3$$
$$\sqrt{15} = 3$$

x =- 3 is not a solution

So, x = 3 is the only solution!

The second type of tasks that we will examine is the equations: $\sqrt{a(x)} \pm \sqrt{b(x)} = c(x)$

First, we must solv inequations $a(x) \geq 0$ and $b(x) \geq 0$, and when we come to form $\sqrt{P(x)} = Q(x)$ then

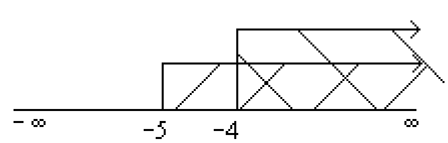
$P(x) = Q(x)^2 \wedge Q(x) \geq 0$ .Obtained solutions"check"in the home equation!

Example 1: Solve the equation: $\sqrt{2x+8} + \sqrt{x+5} = 7$

Solution:

$2x + 8 \geq 0$ and $x + 5 \geq 0$

$x \geq -4$ and $x \geq -5$



$x \in [-4, \infty)$

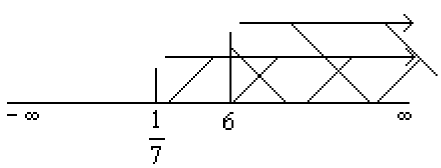
$$\sqrt{2x+8}+\sqrt{x+5}=7\,/()^2$$
$$\sqrt{2x+8}^2+2\sqrt{2x+8}\sqrt{x+5}+\sqrt{x+5}^2=7^2$$
$$2x+8+2\sqrt{(2x+8)(x+5)}+x+5=49$$
$$2\sqrt{(2x+8)(x+5)}=49-2x-8-x-5$$
$$2\sqrt{(2x+8)(x+5)}=36-3x\,/()^2 \rightarrow \quad \text{condition:} \quad 36-3x\ge 0$$
$$4(2x+8)(x+5)=(36-3x)^2 \qquad -3x\ge -36$$
$$4(2x^2+10x+8x+40)=1296-216x+9x^2 \qquad x\le 12$$
$$8x^2+40x+32x+160-1296+216x-9x^2=0$$
$$-x^2+288x-1136=0$$
$$x^2-288x+1136=0$$
$$x_{1,2}=\frac{288\pm 280}{2}$$
$$x_1=284$$
$$x_2=4$$

To remind you on conditions: $x\in[-4,\infty)$ and $x\le 12$,so, **the only solution is x = 4**

Example 2: Solve the equation: $\sqrt{7x-1}-\sqrt{3x-18}=5$

Solution:

$$7x-1\ge 0 \quad \text{i} \quad 3x-18\ge 0$$
$$x\ge\frac{1}{7} \quad \text{i} \quad x\ge 6$$



$$x\in[6,\infty)\rightarrow condition$$

$$\sqrt{7x-1}-\sqrt{3x-18}=5$$

$$\sqrt{7x-1} = 5+\sqrt{3x-18} \ /()^2$$
$$7x-1 = 25+10\sqrt{3x-18}+3x-8$$
$$7x-1-25-3x+18 = 10\sqrt{3x-18}$$
$$4x-8 = 10\sqrt{3x-18} \ /:2$$

$$2x-4 = 5\sqrt{3x-18} \ /()^2 \rightarrow$$ condition : $2x-4 \geq 0$, $x \geq 2$

$$(2x-4)^2 = 25(3x-18)$$
$$4x^2-16x+16 = 75x-450$$
$$4x^2-16x+16-75x+450 = 0$$
$$4x^2-91x+466 = 0$$
$$x_{1,2} = \frac{91\pm\sqrt{825}}{8}$$
$$x_{1,2} = \frac{91\pm5\sqrt{33}}{8}$$
$$x_1 = \frac{91+5\sqrt{33}}{8}$$
$$x_2 = \frac{91-5\sqrt{33}}{8}$$

When this happens, we have to find the approximate value for x_1 and x_2 to see if they satisfy the conditions.

$$x_1 \approx 14{,}97$$
$$x_2 \approx 7{,}78$$

Since the conditions are $x \geq 6$ and $x \geq 2$

We conclude that both solutions are “good”.

Example 3: Solve the equation: $\sqrt{x+3}+\sqrt{x+8} = \sqrt{x+24}$

Solution:

Here we have set 3 conditions:

$x+3 \geq 0$, $x \geq -3$, $x+8 \geq 0$, $x \geq -8$, $x+24 \geq 0$, $x \geq -24$

$$\sqrt{x+3}+\sqrt{x+8}=\sqrt{x+24} \,/()^2$$

$$\sqrt{x+3}^2+2\sqrt{x+3}\sqrt{x+8}+\sqrt{x+8}^2=\sqrt{x+24}^2$$

$$x+3+2\sqrt{(x+3)(x+8)}+x+8=x+24$$

$$2\sqrt{(x+3)(x+8)}=x+24-x-3-x-8$$

$$2\sqrt{(x+3)(x+8)}=13-x \rightarrow \text{ condition: } \quad \begin{array}{l} 13-x\geq 0 \\ -x\geq -13 \\ x\leq 13 \end{array}$$

$$4(x+3)(x+8)=(13-x)^2$$

$$4(x^2+8x+3x+24)=169-26x+x^2$$

$$4x^2+32x+12x+96-169+26x-x^2=0$$

$$3x^2+70x-73=0$$

$$x_{1,2}=\frac{-70\pm\sqrt{5776}}{6}=\frac{-70\pm 76}{6}$$

$$x_1=1$$

$$x_2=-24$$

Whether they are good solutions?

Conditions are $x\geq -3$ and $x\leq 13$, **so x = 1 is only solution**

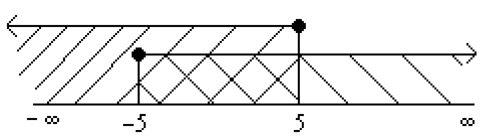
Example 4: Solve the equation: $\sqrt{5+\sqrt[3]{x}}+\sqrt{5-\sqrt[3]{x}}=\sqrt[3]{x}$

Solution:

Here we need to introduce replacement: $\sqrt[3]{x}=t$

$$\sqrt{5+t}+\sqrt{5-t}=t \,/()^2$$

Conditions: $\begin{array}{l} 5+t\geq 0 \\ t\geq -5 \end{array}$ i $\begin{array}{l} 5-t\geq 0 \\ -t\geq -5 \\ t\leq 5 \end{array}$



$$t\in[-5,5]$$

$$\sqrt{5+t}+\sqrt{5-t}=t\ /()^2$$
$$\left(\sqrt{5+t}+\sqrt{5-t}\right)^2=t^2$$
$$\sqrt{5+t}^2+\sqrt{(5+t)-(5-t)}+\sqrt{5-t}^2=t^2$$
$$5+t+2\sqrt{25-t^2}+5-t=t^2$$
$$2\sqrt{25-t^2}=t^2-10\ /()^2 \rightarrow \text{ condition: } t^2-10\geq 0$$

$$4(25-t^2)=(t^2-10)^2$$
$$4(25-t^2)=t^4-20t^2+100$$
$$100-4t^2=t^4-20t^2+100$$
$$t^4-16t^2=0$$
$$t^2(t^2-16)=0$$
$$t^2=0 \Rightarrow t=0$$
$$t^2-16=0 \Rightarrow t=+4, t=-4$$

for $t=4 \Rightarrow \sqrt[3]{x}=4 \Rightarrow x=64$ is solution
for $t=-4 \Rightarrow \sqrt[3]{x}-4 \Rightarrow x=-64$ is not solution
for $t=0 \Rightarrow x=0$ not solution

So: $x=64$ is only solution!!!