

Exponentiation

$$a \underset{n\text{-times}}{*} a \underset{n\text{-times}}{*} \dots \underset{n\text{-times}}{*} a = a^n$$

If $a \in \mathbb{R} \wedge a \neq 0$ and $n \in \mathbb{N}$ then:

By definition is:

$$1) a^0 = 1 \rightarrow \text{example: } 5^0 = 1, (-3)^0 = 1, \left(\frac{4}{7}\right)^0 = 1$$

$$2) a^{-n} = \frac{1}{a^n} \rightarrow \text{example: } 3^{-2} = \frac{1}{3^2} = \frac{1}{9}, 5^{-3} = \frac{1}{5^3} = \frac{1}{125}$$

More rules:

$$3) a^m \cdot a^n = a^{m+n} \rightarrow \text{example: } 3^2 \cdot 3^5 = 3^{2+5} = 3^7$$

$$4) a^m : a^n = a^{m-n} \rightarrow \text{example: } 7^{10} : 7^6 = 7^{10-6} = 7^4$$

$$5) (a^m)^n = a^{m \cdot n} \rightarrow \text{example: } (2^3)^5 = 2^{3 \cdot 5} = 2^{15}$$

$$6) (a \cdot b)^n = a^n \cdot b^n \rightarrow \text{example: } (12 \cdot 11)^5 = 12^5 \cdot 11^5$$

$$7) \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \rightarrow \text{example: } \left(\frac{7}{4}\right)^2 = \frac{7^2}{4^2}$$

$$8) \left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n \rightarrow \text{example: } \left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^2 = \frac{3^2}{2^2} = \frac{9}{4}$$

We should pay attention to the record: $(-5)^2 = (-5)(-5) = 25$, while $-5^2 = -5 \cdot 5 = -25$.

In general, is:

$$(-a)^{2k} = a^{2k}$$

$$(-a)^{2k+1} = -a^{2k+1}$$

EXAMPLES:

1) Simplify: $\frac{(2^7 : 2^5) \cdot 2^3}{2^4 : 2^2}$

$$\frac{(2^7 : 2^5) \cdot 2^3}{2^4 : 2^2} = \frac{2^{7-5} \cdot 2^3}{2^{4-2}} = \frac{2^2 \cdot 2^3}{2^2} = \frac{2^{2+3}}{2^2} = \frac{2^5}{2^2} = 2^{5-2} = 2^3 = 8$$

2) Simplify: $\frac{3^5 \cdot 9^3}{27^2 \cdot 3}$

$$\frac{3^5 \cdot 9^3}{27^2 \cdot 3} = \frac{3^5 \cdot (3^2)^3}{(3^3)^2 \cdot 3^1} = \frac{3^5 \cdot 3^6}{3^6 \cdot 3^1} = \frac{3^5}{3^1} = 3^{5-1} = 3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81$$

3) Simplify: $\frac{(x^4)^3 \cdot x^3 : x^5}{(x^5 : x^2)^3}$

$$\frac{(x^4)^3 \cdot x^3 : x^5}{(x^5 : x^2)^3} = \frac{x^{12} \cdot x^3 : x^5}{(x^{5-2})^3} = \frac{x^{12+3-5}}{(x^3)^3} = \frac{x^{10}}{x^9} = x^{10-9} = x^1 = x$$

4) Simplify: $\frac{3^{n+1} \cdot 3^{n+2}}{3^{2n+4}}$

$$\begin{aligned} \frac{3^{n+1} \cdot 3^{n+2}}{3^{2n+4}} &= \frac{3^{n+1+n+2}}{3^{2n+4}} = \frac{3^{2n+3}}{3^{2n+4}} = \\ &= 3^{(2n+3)-(2n+4)} = 3^{2n+3-2n-4} = 3^{-1} = \frac{1}{3^1} = \frac{1}{3} \end{aligned}$$

5) Simplify : $0,5^{-1} + 0,25^{-2} + 0,125^{-3} + 0,0625^{-4}$

$$0,5^{-1} + 0,25^{-2} + 0,125^{-3} + 0,0625^{-4} =$$

$$\left(\frac{1}{2}\right)^{-1} + \left(\frac{1}{4}\right)^{-2} + \left(\frac{1}{8}\right)^{-3} + \left(\frac{1}{16}\right)^{-4} =$$

$$\left(\frac{2}{1}\right)^1 + \left(\frac{4}{1}\right)^2 + \left(\frac{8}{1}\right)^3 + \left(\frac{16}{1}\right)^4 =$$

$$2^1 + 4^2 + 8^3 + 16^4 = 2 + 16 + 512 + 65536 = 66066$$

6) $1^{-1} + 2^{-2} + 3^{-3} + (-1)^{-1} + (-2)^{-2} + (-3)^{-3} = ?$

$$\begin{aligned} & 1^{-1} + 2^{-2} + 3^{-3} + (-1)^{-1} + (-2)^{-2} + (-3)^{-3} = \\ & \frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{(-1)^1} + \frac{1}{(-2)^2} + \frac{1}{(-3)^3} = \\ & 1 + \frac{1}{4} + \frac{1}{27} - 1 + \frac{1}{4} - \frac{1}{27} = \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2} \end{aligned}$$

7) If $a = 5^3 \cdot \left(\frac{1}{4}\right)^{-4} \cdot \left(\frac{3}{2}\right)^2$ and $b = 10^3 \left(\frac{5}{3}\right)^{-2}$ find $a \cdot b^{-1} = ?$

$$\begin{aligned} a &= 5^3 \cdot \left(\frac{1}{4}\right)^{-4} \cdot \left(\frac{3}{2}\right)^2 = 5^3 \cdot \left(\frac{4}{1}\right)^4 \cdot \frac{3^2}{2^2} = \frac{5^3 \cdot 4^4 \cdot 3^2}{2^2} \\ &= \frac{5^3 \cdot (2^2)^4 \cdot 3^2}{2^2} = 5^3 \cdot (2^2)^3 \cdot 3^2 = 5^3 \cdot 2^6 \cdot 3^2 \end{aligned}$$

$$b = 10^3 \cdot \left(\frac{5}{3}\right)^{-2} = 10^3 \cdot \left(\frac{3}{5}\right)^2 = \frac{10^3 \cdot 3^2}{5^2} = \frac{5^3 \cdot 2^3 \cdot 3^2}{5^2} = 5 \cdot 2^3 \cdot 3^2$$

Finally: calculate $a \cdot b^{-1}$

$$a \cdot b^{-1} = 5 \cdot 2^3 \cdot 3^2 \cdot \frac{1}{5 \cdot 2^3 \cdot 3^2} = 5^2 \cdot 2^3 = 25 \cdot 8 = 200$$

8) $\left(\left(\frac{5x^{-5}}{2y^{-2}} \right)^{-2} \cdot \left(\frac{y^{-1}}{5x^{-1}} \right)^{-3} \right) : 10x^2y^{-3} = ?$

$$\begin{aligned} & \left(\left(\frac{5x^{-5}}{2y^{-2}} \right)^{-2} \cdot \left(\frac{y^{-1}}{5x^{-1}} \right)^{-3} \right) : 10x^2y^{-3} = \\ & \left(\frac{5^{-2} \cdot x^{10}}{2^{-2} \cdot y^4} \cdot \frac{y^3}{5^{-3} \cdot x^3} \right) : 10x^2y^{-3} = \\ & (5^{-2+3} \cdot x^{10-3} \cdot y^{3-4} \cdot 2^2) : 10x^2y^{-3} = \\ & (5^1 \cdot x^7 \cdot y^{-1} \cdot 4) : 10x^2y^{-3} = \\ & \frac{20}{10} x^{7-2} y^{-1-(-3)} = 2x^5y^{-1+3} = 2x^5y^2 \end{aligned}$$

9) If $10^x = \frac{\frac{1}{2}10^{-3} + \frac{1}{2}10^{-4}}{55 \cdot 10^{-7}}$ Find $x=?$

$$\frac{\frac{1}{2}10^{-3} + \frac{1}{2}10^{-4}}{55 \cdot 10^{-7}} = \frac{\frac{1}{2000} + \frac{1}{20000}}{55} =$$

$$\frac{\frac{1}{2000} \left(1 + \frac{1}{10}\right)}{55} = \frac{\frac{1}{2000} \cdot \frac{11}{10}}{55} = \frac{\frac{11}{20000}}{55} =$$

$$\frac{\frac{11 \cdot 10000000}{20000 \cdot 55}}{\frac{10000000}{100000}} = \frac{10000000}{100000} = 100 = 10^2$$

Now $10^x = 10^2$, then $x = 2$

10)

a) $A \cdot 10^{-5} = 0,2 \cdot 0,008$

b) $B \cdot 10^{-6} = 0,04 \cdot 0,006$

A = ?

$$A \cdot 10^{-5} = 0,2 \cdot 0,008$$

$$A \cdot 10^{-5} = 2 \cdot 10^{-1} \cdot 8 \cdot 10^{-3}$$

$$A \cdot 10^{-5} = 16 \cdot 10^{-4}$$

$$A = \frac{16 \cdot 10^{-4}}{10^{-5}}$$

$$A = 16 \cdot 10^{-4-(-5)}$$

$$A = 16 \cdot 10^{-4+5}$$

$$A = 16 \cdot 10$$

$$A = 160$$

B = ?

$$B \cdot 10^{-6} = 0,04 \cdot 0,006$$

$$B \cdot 10^{-6} = 4 \cdot 10^{-2} \cdot 6 \cdot 10^{-3}$$

$$B \cdot 10^{-6} = 24 \cdot 10^{-5}$$

$$B = \frac{24 \cdot 10^{-5}}{10^{-6}}$$

$$B = 24 \cdot 10^{-5+6}$$

$$B = 24 \cdot 10$$

$$B = 240$$

Here we use a real number in the recording system with the 10.
This is a good options when the number is very big.

For example:

1) speed of light is approximately $c = 300000000m/s$, "easier" notes is $c = 3 \cdot 10^8 m/s$

2) $\frac{1}{500000} = \frac{1}{5 \cdot 10^5} = \frac{1}{5} \cdot 10^{-5} = \frac{2}{10} \cdot 10^{-5} = 2 \cdot 10^{-1} \cdot 10^{-5} = 2 \cdot 10^{-6}$

3) $0,000069 = 6,9 \cdot 10^{-5} \approx 7 \cdot 10^{-5}$

4) The surface of the earth is $510083000km^2$ but we note: $\approx 5 \cdot 10^8 km^2$

$$11) \left(\frac{3a^{-x}}{1-a^{-x}} - \frac{2a^{-x}}{1+a^{-x}} - \frac{a^x}{a^{2x}-1} \right) : \frac{a^{-x}}{a^x - a^{-x}} = ?$$

$$\begin{aligned} & \left(\frac{3a^{-x}}{1-a^{-x}} - \frac{2a^{-x}}{1+a^{-x}} - \frac{a^x}{a^{2x}-1} \right) : \frac{a^{-x}}{a^x - a^{-x}} = \\ & \left(\frac{\frac{3}{a^x}}{1-\frac{1}{a^x}} - \frac{\frac{2}{a^x}}{1+\frac{1}{a^x}} - \frac{a^x}{a^{2x}-1} \right) : \frac{\frac{1}{a^x}}{a^x - \frac{1}{a^x}} = \\ & \left(\frac{\frac{3}{a^x}}{\frac{a^x-1}{a^x}} - \frac{\frac{2}{a^x}}{\frac{a^x+1}{a^x}} - \frac{a^x}{a^{2x}-1} \right) : \frac{\frac{1}{a^x}}{\frac{a^{2x}-1}{a^x}} = \\ & \left(\frac{3}{a^x-1} - \frac{2}{a^x+1} - \frac{a^x}{(a^x-1)(a^x+1)} \right) : \frac{1}{a^{2x}-1} = \\ & \frac{3(a^x+1) - 2(a^x-1) - a^x}{(a^x-1)(a^x+1)} \cdot \frac{a^{2x}-1}{1} = \\ & \frac{3a^x+3-2a^x+2-a^x}{(a^x-1)(a^x+1)} \cdot \frac{(a^x-1)(a^x+1)}{1} = \\ & = 3+2=5 \end{aligned}$$

$$12) \left(\frac{x-x^{-2}}{x^{-2}+x^{-1}+1} - \frac{x-x^{-1}}{1+x^{-2}+2 \cdot x^{-1}} \right) : \frac{1-x^{-1}}{1+x^{-1}} = ?$$

$$\begin{aligned} & \left(\frac{x-x^{-2}}{x^{-2}+x^{-1}+1} - \frac{x-x^{-1}}{1+x^{-2}+2 \cdot x^{-1}} \right) : \frac{1-x^{-1}}{1+x^{-1}} = \\ & \left(\frac{x-\frac{1}{x^2}}{\frac{1}{x^2}+\frac{1}{x}+1} - \frac{x-\frac{1}{x}}{1+\frac{1}{x^2}+\frac{2}{x}} \right) : \frac{1-\frac{1}{x}}{1+\frac{1}{x}} = \\ & \left(\frac{\frac{x^3-1}{x^2}}{\frac{1+x+x^2}{x^2}} - \frac{\frac{x^2-1}{x}}{x^2+1+2x} \right) : \frac{\frac{x-1}{x}}{\frac{x+1}{x}} = \\ & \left(\frac{(x-1)(x^2+x+1)}{x^2+x+1} - \frac{x(x-1)(x+1)}{(x+1)^2} \right) : \frac{x-1}{x+1} = \\ & \left(\frac{x-1}{1} - \frac{x(x-1)}{x+1} \right) : \frac{x-1}{x+1} = \\ & \frac{(x-1)(x+1) - x(x-1)}{x+1} \cdot \frac{x+1}{x-1} = \\ & \frac{(x-1)[(x+1)-x]}{x-1} = x+1-x=1 \end{aligned}$$

$$\begin{aligned}
13) \quad A &= \left(\frac{a^n}{1-a^{-n}} + \frac{a^{-n}}{1+a^{-n}} \right) - \left(\frac{a^n}{1+a^{-n}} + \frac{a^{-n}}{1-a^{-n}} \right) = ? \\
A &= \left(\frac{a^n}{1-a^{-n}} + \frac{a^{-n}}{1+a^{-n}} \right) - \left(\frac{a^n}{1+a^{-n}} + \frac{a^{-n}}{1-a^{-n}} \right) \\
A &= \left(\frac{a^n}{1-\frac{1}{a^n}} + \frac{\frac{1}{a^n}}{1+\frac{1}{a^n}} \right) - \left(\frac{a^n}{1+\frac{1}{a^n}} + \frac{\frac{1}{a^n}}{1-\frac{1}{a^n}} \right) \\
A &= \left(\frac{\frac{a^n}{1}}{\frac{a^n-1}{a^n}} + \frac{\frac{1}{a^n}}{\frac{a^n+1}{a^n}} \right) - \left(\frac{a^n}{\frac{a^n+1}{a^n}} + \frac{\frac{1}{a^n}}{\frac{a^n-1}{a^n}} \right) \\
A &= \left(\frac{a^{2n}}{a^n-1} + \frac{1}{a^n+1} \right) - \left(\frac{a^{2n}}{a^n+1} + \frac{1}{a^n-1} \right) \\
A &= \frac{a^{2n}(a^n+1)+1(a^n-1)}{(a^n-1)(a^n+1)} - \frac{a^{2n}(a^n-1)+1(a^n+1)}{(a^n+1)(a^n-1)} \\
A &= \frac{a^{3n}+a^{2n}+a^n-1-(a^{3n}-a^{2n}+a^n+1)}{(a^n-1)(a^n+1)} \\
A &= \frac{a^{3n}+a^{2n}+a^n-1-a^{3n}+a^{2n}-a^n-a}{(a^n-1)(a^n+1)} \\
A &= \frac{2a^{2n}-2}{(a^n-1)(a^n+1)} = \frac{2(a^n-1)(a^n+1)}{(a^n-1)(a^n+1)} = 2
\end{aligned}$$