SOME IMPORTANT INEQUALITY:

1) $x^2 \geq 0$ for all $x \in R$

Square a term is always positive or equal to zero (for x = 0)

Examples: $\rightarrow\ x^2 + 4x + 4 = (x+2)^2 \geq 0$ for $\forall x \in R$

$\rightarrow\ -a^2 + 2a - 1 = -(a-1)^2 \leq 0$ for $\forall a \in R$

$\rightarrow\ x^2 - xy + y^2 \geq 0$ because

$$x^2 - xy + \left(\frac{y}{2}\right)^2 - \left(\frac{y}{2}\right)^2 + y^2 = \left(x - \frac{y}{2}\right)^2 - \frac{y^2}{4} + y^2 = \left(x - \frac{y}{2}\right)^2 + \frac{3y^2}{4}$$

$\left(x - \frac{y}{2}\right)^2 \geq 0$ and $\frac{3y^2}{4} \geq 0$,

2) $$\frac{x^2 + y^2 + z^2 + 3}{2} \geq x + y + z$$

Proof:

$$\begin{matrix} (x-1)^2 \geq 0 \\ (y-1)^2 \geq 0 \\ (z-1)^2 \geq 0 \end{matrix} \quad \Rightarrow \quad \begin{aligned} &(x-1)^2 + (y-1)^2 + (z-1)^2 \geq 0 \\ &x^2 - 2x + 1 + y^2 - 2y + 1 + z^2 - 2z + 1 \geq 0 \\ &x^2 + y^2 + z^2 + 3 \geq 2x + 2y + 2z \\ &x^2 + y^2 + z^2 + 3 \geq 2(x + y + z) \\ &\frac{x^2 + y^2 + z^2 + 3}{2} \geq x + y + z \end{aligned}$$

3) Demonstrate that $\forall a > 0 \ \Rightarrow\ a + \frac{1}{2} \geq 2$

Proof:

$$(a-1)^2 \geq 0$$
$$a^2 - 2a + 1 \geq 0$$
$$a^2 + 1 \geq 2a \,/ : a$$
$$a + \frac{1}{a} \geq 2$$

4) Prove that $\forall x \geq 0$ i $\forall y \geq 0$ $\sqrt{xy} \leq \frac{x+y}{2}$ (geometric mean $\leq$ arithmetical mean)

Proof:

$$\left(\sqrt{x}+\sqrt{y}\right)^2 \geq 0$$
$$\sqrt{x}^2 - 2\sqrt{x}\sqrt{y} + \sqrt{y}^2 \geq 0$$
$$x - 2\sqrt{xy} + y \geq 0$$
$$x + y = 2\sqrt{xy} \;/:2$$
$$\frac{x+y}{2} \geq \sqrt{xy}$$

Of course, equality is if x = y.

5) Demonstrate that: $\forall x, y, z \ \ (0 \leq x, 0 \leq y, 0 \leq z) \quad \Rightarrow \quad \sqrt[3]{xyz} \leq \frac{a^3+b^3+c^3}{3}$

Proof: Introduce first:

$$x = a^3$$
$$y = b^3$$
$$z = c^3$$

$$\sqrt[3]{xyz} \leq \frac{a^3+b^3+c^3}{3}$$
$$3abc \leq a^3+b^3+c^3$$
$$a^3+b^3+c^3-3abc \geq 0$$

How is : $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ac)$

here is $a+b+c \geq 0$ certainly, because $0 \leq x, 0 \leq y, 0 \leq z$

and $a^2+b^2+c^2-ab-bc-ac = \frac{1}{2}\left[(a-b)^2+(b-c)^2+(c-a)^2\right] \geq 0$

So the product of two such expression is> 0 and is: $abc \leq \frac{a^3+b^3+c^3}{3}$

Take heed: = character is if x=y=z