

## Operations with rational algebraic expressions

Greatest common divisor, and least common container polynomial

**GCD** polynomial P and Q is a polynomial D, which has the highest level among polynomials, which are dividers and polynomial P and Q.

**LCC** polynomial P and Q is a polynomial S, who has the lowest level among polynomials, which are divisible and polynomials P and Q.

Example 1: Find S and D for polynomials:

$$P(x) = x^2 - 4$$

$$Q(x) = x^2 - x - 2$$

$$R(x) = x^2 - 3x + 2$$

First, we must each of them apart on the facts (of course, the use of the procedure mentioned in the previous chapter).

$$P(x) = x^2 - 4 = x^2 - 2^2 = (x-2)(x+2)$$

$$Q(x) = x^2 - x - 2 = x^2 - 2x + x - 2 = x(x-2) + 1(x-2) = (x-2)(x+1)$$

$$R(x) = x^2 - 3x + 2 = x^2 - x - 2x + 2 = x(x-1) - 2(x-1) = (x-1)(x-2)$$

**D** is actually “one”, which it has in each of polynomials.

It is obvious x-2. So:

$$\text{GCD} = x-2 \quad \text{or} \quad D = x-2$$

S is “Union”. It must be divisible by all three polynomials. So:

$$S = (x-2)(x+2)(x-1)(x+1)$$

Example 2: Find S and D for polynomials:

$$P = a^2 - ab$$

$$Q = a^2 - b^2$$

$$R = a^2 - 2ab + b^2$$

$$P = a^2 - ab = a(a-b)$$

$$Q = a^2 - b^2 = (a-b)(a+b)$$

$$R = a^2 - 2ab + b^2 = (a-b)^2$$

$D = (a - b) \rightarrow$  because it has all three

$S = a(a - b)^2(a + b) \rightarrow$  divisible by all three

**Example 3:** Find S and D for polynomials:

$$A = x^2 - xy$$

$$B = xy + y^2$$

$$\begin{aligned} A &= x(x - y) \\ B &= y(x + y) \end{aligned} \Rightarrow \text{NZS} = xy(x - y)(x + y)$$

What will be D? There are no factors that are contained in A and B. In such a situation  $\text{NZD} = 1$ , and we say that polynomials are the mutual primes.

**Example 4: Find S...**

$$9a + 15 = 3(3a + 5)$$

$$36a^2 - 100 = 4(9a^2 - 25) = 4(3a - 5)(3a + 5)$$

$$-9a^2 + 30a - 25 = -(9a^2 - 30a + 25) = -(3a - 5)^2$$

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$$S = -12(3a + 5)(3a - 5)$$

**Example 5: Find S**

$$4a^2 + 4ab + b^2 = (2a + b)^2$$

$$4a^2 - b^2 = (2a - b)(2a + b)$$

$$8a^3 + b^3 = (2a)^3 + b^3 = (2a + b)(4a^2 - 2ab + b^2)$$

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$$S = (2a + b)^2(2a - b)(4a^2 - 2ab + b^2)$$

**Example 6:**

$$3x^3 - 12x^2 + 12x = 3x(x^2 - 4x + 4) = 3x(x - 2)^2$$

$$5x^4 + 20x^3 + 20x^2 = 5x^2(x^2 + 4x + 4) = 5x^2(x + 2)^2$$

$$3nx^2 - 12n = 3n(x^2 - 4) = 3n(x - 2)(x + 2)$$

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$$S = 15nx^2(x - 2)^2(x + 2)^2$$

### Example 7:

$$2a^4 - 2 = 2(a^4 - 1) = 2(a^2 - 1)(a^2 + 1) = 2(a - 1)(a + 1)(a^2 + 1)$$

$$a^3 + a^2 + a + 1 = a^2(a + 1) + 1(a + 1) = (a + 1)(a^2 + 1)$$

$$a^3 - a^2 + a + 1 = a^2(a - 1) + 1(a - 1) = (a - 1)(a^2 + 1)$$

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$$S = 2(a - 1)(a + 1)(a^2 + 1)$$

### How to use S?

1) Simplify:  $\frac{a}{ab - b^2} + \frac{b}{a^2 - ab} - \frac{a + b}{ab} =$

$$\frac{a}{ab - b^2} + \frac{b}{a^2 - ab} - \frac{a + b}{ab} = \text{first every denominator apart on factors} =$$

$$\frac{a}{b(a - b)} + \frac{b}{a(a - b)} - \frac{a + b}{ab} = \text{then find S, it is } ab(a - b), \text{ make fractional expansion.}$$

$$= \frac{a \cdot a + b \cdot b - (a + b)(a - b)}{ab(a - b)} =$$

$$= \frac{a^2 + b^2 - (a^2 - b^2)}{ab(a - b)} = \frac{a^2 + b^2 - a^2 + b^2}{ab(a - b)} = \frac{2b^2}{ab(a - b)} = \frac{2b}{a(a - b)}$$

Before the beginning of the **conditions** should be set tasks. Since sharing zero is not allowed :

$$a \neq 0; \quad b \neq 0; \quad a - b \neq 0 \Rightarrow a \neq b$$

2) Simplify:  $\frac{1}{x^2 - x} + \frac{2}{1 - x^2} + \frac{1}{x^2 + x} =$

$$\frac{1}{x^2 - x} + \frac{2}{1 - x^2} + \frac{1}{x^2 + x} = \frac{1}{x(x - 1)} + \frac{2}{(1 - x)(1 + x)} + \frac{1}{x(x + 1)} = \text{what's the problem?}$$

Phrases  $(1 + x)$  and  $(x + 1)$  are not, because we have commutative law  $(A + B = B + A)$ , but express

$(x - 1)$  and  $(1 - x)$  are! This problem will be solved by law :  $A - B = - (B - A)$

$$= \frac{1}{x(x - 1)} - \frac{2}{(x - 1)(1 + x)} + \frac{1}{x(x + 1)} =$$

$$= \frac{1 \cdot (x + 1) - 2(x) + 1(x - 1)}{x(x - 1)(x + 1)}$$

$$= \frac{x + 1 - 2x + x - 1}{x(x - 1)(x + 1)}$$

$$= \frac{0}{x(x - 1)(x + 1)} = 0$$

Of course, the task **conditions** are:

$$x \neq 0; x-1 \neq 0 \Rightarrow x \neq 1; x+1 \neq 0 \Rightarrow x \neq -1$$

$$3) \frac{a+1}{a+2} + \frac{6a}{a^2-4} - \frac{2a-1}{a-2} = ?$$

$$\begin{aligned} & \frac{a+1}{a+2} + \frac{6a}{a^2-4} - \frac{2a-1}{a-2} = \\ & \frac{a+1}{a+2} + \frac{6a}{(a-2)(a+2)} - \frac{2a-1}{a-2} = \\ & \frac{(a+1)(a-2) + 6a - (2a-1)(a+2)}{(a-2)(a+2)} = \\ & \frac{(a^2 - 2a + a - 2) + 6a - (2a^2 + 4a - a - 2)}{(a-2)(a+2)} = \\ & \frac{a^2 - 2a + a - 2 + 6a - 2a^2 - 4a + a + 2}{(a-2)(a+2)} = \\ & \frac{-a^2 + 2a}{(a-2)(a+2)} = \frac{-a(a-2)}{(a-2)(a+2)} = \frac{-a}{a+2} \end{aligned}$$

**conditions** of the task are:  $a+2 \neq 0 \Rightarrow a \neq -2$   
 $a-2 \neq 0 \Rightarrow a \neq 2$

$$4) \frac{x}{x-1} - \frac{3x-1}{x-2} + \frac{2x+1}{x^2-3x+2} = ?$$

$$\frac{x}{x-1} - \frac{3x-1}{x-2} + \frac{2x+1}{x^2-3x+2} =$$

$$[[[x^2 - 3x + 2 = x^2 - 2x - x + 2 = x(x-2) - 1(x-2) = (x-2)(x-1) ]]]$$

$$\begin{aligned} & \frac{x}{x-1} - \frac{3x-1}{x-2} + \frac{2x+1}{(x-2)(x-1)} = \\ & \frac{x(x-2) - (3x-1)(x-1) + 1(2x+1)}{(x-1)(x-2)} = \\ & \frac{x^2 - 2x - (3x^2 - 3x - x + 1) + 2x + 1}{(x-1)(x-2)} = \\ & \frac{x^2 - 2x - 3x^2 + 3x + x - 1 + 2x + 1}{(x-1)(x-2)} = \\ & \frac{-2x^2 + 4x}{(x-1)(x-2)} = \\ & \frac{-2x(x-2)}{(x-1)(x-2)} = \frac{-2x}{x-1} \end{aligned}$$

**conditions** of the task are:

$$x - 1 \neq 0 \Rightarrow x \neq 1$$

$$x - 2 \neq 0 \Rightarrow x \neq 2$$

$$5) \frac{1}{x^2 + 10x + 25} + \frac{1}{x^2 - 10x + 25} + \frac{2}{x^2 - 25} = ?$$

$$\begin{aligned} & \frac{1}{x^2 + 10x + 25} + \frac{1}{x^2 - 10x + 25} + \frac{2}{x^2 - 25} = \\ & \frac{1}{(x+5)^2} + \frac{1}{(x-5)^2} + \frac{2}{(x-5)(x+5)} = \\ & \frac{1 \cdot (x-5)^2 + 1 \cdot (x+5)^2 + 2 \cdot (x^2 - 25)}{(x+5)^2(x-5)^2} = \\ & \frac{x^2 - 10x + 25 + x^2 + 10x + 25 + 2(x^2 - 25)}{(x+5)^2(x-5)^2} = \\ & \frac{2x^2 + 50 + 2x^2 - 50}{(x+5)^2(x-5)^2} = \\ & = \frac{4x^2}{(x+5)^2(x-5)^2} = \frac{4x^2}{(x^2 - 25)^2} \end{aligned}$$

**conditions** of the task are:  $x + 5 \neq 0 \Rightarrow x \neq -5$   
 $x - 5 \neq 0 \Rightarrow x \neq 5$

Multiplication and sharing of rational algebraic expression is, as well as in ordinary fraction, with the first to have "every" apart on the facts. So:

$$\frac{A}{B} \cdot \frac{C}{D} = \frac{A \cdot C}{B \cdot D} \quad \text{and} \quad \frac{A}{B} : \frac{C}{D} = \frac{A}{B} \cdot \frac{D}{C}$$

$$1) \text{ Simplify: } \frac{a^2 - a}{a^2 - 1} \cdot \frac{a^2 + 2a + 1}{a^2 + a} =$$

$$\frac{a^2 - a}{a^2 - 1} \cdot \frac{a^2 + 2a + 1}{a^2 + a} = \frac{a(a-1)}{(a-1)(a+1)} \cdot \frac{(a+1)^2}{a(a+1)} = \frac{1}{1} \cdot \frac{1}{1} = 1$$

$$a^2 - 1 \neq 0 \quad \text{i} \quad a^2 + a \neq 0 \\ a \neq 1, a \neq -1, a \neq 0$$

$$2) \text{ Simplify: } \frac{a^2 - ab}{a^2 + ab} \cdot \frac{a^2b + ab^2}{ab} = ?$$

$$\frac{a^2 - ab}{a^2 + ab} \cdot \frac{a^2b + ab^2}{ab} = \frac{a(a-b)}{a(a+b)} \cdot \frac{ab(a+b)}{ab} = \frac{a-b}{1} = a-b \quad a \neq 0, b \neq 0, a+b \neq 0$$

$$3) \frac{x^2 - 25}{x^2 - 3x} : \frac{x^2 + 5x}{x^2 - 9} = ?$$

$$\frac{x^2 - 25}{x^2 - 3x} : \frac{x^2 + 5x}{x^2 - 9} = \frac{(x-5)(x+5)}{x(x-3)} : \frac{x(x+5)}{(x-3)(x+3)} = \frac{(x-5)(x+5)}{x(x-3)} \cdot \frac{(x-3)(x+3)}{x(x+5)} = \frac{(x-5) \cdot (x+3)}{x^2}$$

**conditions** of the task are:  $x \neq 0, x-3 \neq 0, x+3 \neq 0$   
 $x \neq 3, x \neq -3$

$$4) \frac{a^2 + b^2}{1 + 2m + m^2} : \frac{a^4 - b^4}{1 - 2m^2 + m^4} = ?$$

$$\frac{a^2 + b^2}{1 + 2m + m^2} : \frac{a^4 - b^4}{1 - 2m^2 + m^4} =$$

$$\frac{a^2 + b^2}{(1+m)^2} : \frac{(a^2 - b^2)(a^2 + b^2)}{(1-m^2)^2} =$$

$$\frac{a^2 + b^2}{(1+m)^2} \cdot \frac{(1-m)^2(1+m)^2}{(a-b)(a+b)(a^2 + b^2)} = \frac{(1-m)^2}{(a-b)(a+b)}$$

$x \neq b, x \neq -b, m \neq 1, x \neq -1,$

$$5) \frac{a^2 + b^2 - c^2 + 2ab}{a^2 + c^2 - b^2 + 2ac} = ?$$

$$\frac{a^2 + b^2 - c^2 + 2ab}{a^2 + c^2 - b^2 + 2ac} = \frac{a^2 + 2ab + b^2 - c^2}{a^2 + 2ac + c^2 - b^2} = \frac{(a+b)^2 - c^2}{(a+c)^2 - b^2} = \frac{(a+b-c)(a+b+c)}{(a+c-b)(a+c+b)} = \frac{a+b-c}{a+c-b}$$

$a+c-b \neq 0$  and  $a+c+b \neq 0$

$$6) \frac{x^2 - 5x + 6}{x^2 - 3x + 2} = ?$$

$$\frac{x^2 - 5x + 6}{x^2 - 3x + 2} = \frac{x^2 - 3x - 2x + 6}{x^2 - 2x - x + 2} =$$

$$= \frac{x(x-3) - 2(x-3)}{x(x-2) - 1(x-2)} = \frac{(x-3)(x-2)}{(x-2)(x-1)} = \frac{x-3}{x-1}$$

$x-2 \neq 0, x-1 \neq 0$

$$7) \left( \frac{x}{y^2 + xy} - \frac{2}{x+y} + \frac{y}{x^2 + xy} \right) : \left( \frac{x}{y} - 2 + \frac{y}{x} \right) = ?$$

$$\left( \frac{x}{y^2 + xy} - \frac{2}{x+y} + \frac{y}{x^2 + xy} \right) : \left( \frac{x}{y} - 2 + \frac{y}{x} \right) =$$

$$\left(\frac{x}{y(y+x)} - \frac{2}{x+y} + \frac{y}{x(x+y)}\right) : \left(\frac{x^2 - 2xy + y^2}{xy}\right) =$$

$$\frac{x^2 - 2xy + y^2}{xy(x+y)} : \frac{x^2 - 2xy + y^2}{xy} =$$

$$\frac{(x-y)^2}{xy(x+y)} \cdot \frac{xy}{(x-y)^2} = \frac{1}{x+y}$$

$$x \neq 0, y \neq 0, x+y \neq 0, x-y \neq 0$$

$$8) \left(\frac{a}{6-3a} + \frac{a}{a+2} + \frac{4a}{a^2-4}\right) : \frac{a-4}{a-2} = ?$$

$$\left(\frac{a}{6-3a} + \frac{a}{a+2} + \frac{4a}{a^2-4}\right) : \frac{a-4}{a-2} =$$

$$\left(\frac{a}{3(2-a)} + \frac{a}{a+2} + \frac{4a}{(a-2)(a+2)}\right) : \frac{a-4}{a-2} =$$

$$\left(\frac{-a}{3(a-2)} + \frac{a}{a+2} + \frac{4a}{(a-2)(a+2)}\right) \cdot \frac{a-2}{a-4} =$$

$$\frac{-a(a+2) + 3a(a-2) + 12a}{3(a-2)(a+2)} \cdot \frac{a-2}{a-4} =$$

$$\frac{-a^2 - 2a + 3a^2 - 6a + 12a}{3(a+2)(a-4)} =$$

$$\frac{2a^2 + 4a}{3(a+2)(a-4)} = \frac{2a(a+2)}{3(a+2)(a-4)} = \frac{2a}{3(a-4)}$$

$$9) \frac{1}{1-x} + \frac{1}{1+x} + \frac{2}{1+x^2} + \frac{4}{1+x^4} + \frac{8}{1+x^8} + \frac{16}{1+x^{16}} = ?$$

This task can not solve the "classic", try to gather the first two:

$$\frac{1}{1-x} + \frac{1}{1+x} = \frac{1+x+1-x}{(1-x)(1+x)} = \frac{2}{1-x^2}$$

Adding his third addend:

$$\frac{2}{1-x^2} + \frac{2}{1+x^2} = \frac{2(1+x^2) + 2(1-x^2)}{(1-x^2)(1+x^2)} = \frac{2+2x^2+2-2x^2}{1-x^4} = \frac{4}{1-x^4}$$

This works!

$$\frac{4}{1-x^4} + \frac{4}{1+x^4} = \frac{4+4x^2+4-4x^2}{(1-x^4)(1+x^2)} = \frac{8}{1-x^8}$$

Go further...

$$\frac{8}{1-x^8} + \frac{8}{1+x^8} = \frac{8+8x^8+8-8x^8}{(1-x^8)(1+x^8)} = \frac{16}{1-x^{16}}$$

Finally:

$$\frac{16}{1-x^{16}} + \frac{16}{1+x^{16}} = \frac{16+16x^{16}+16-16x^{16}}{(1-x^{16})(1+x^{16})} = \frac{32}{1-x^{32}} \quad 1-x \neq 0 \text{ i } 1+x \neq 0$$

$$\frac{1}{1-x} + \frac{1}{1+x} + \frac{2}{1+x^2} + \frac{4}{1+x^4} + \frac{8}{1+x^8} + \frac{16}{1+x^{16}} = \frac{32}{1-x^{32}}$$

10) Prove that the value of expression  $\frac{4}{a + \frac{1}{b + \frac{1}{c}}} : \frac{1}{a + \frac{1}{b}} - \frac{4}{b(abc + a + c)}$  does not depend on a, b, c and d.

$$\frac{4}{a + \frac{1}{b + \frac{1}{c}}} : \frac{1}{a + \frac{1}{b}} - \frac{4}{b(abc + a + c)} = \frac{4}{a + \frac{1}{\frac{bc+1}{c}}} : \frac{1}{a + \frac{1}{b}} - \frac{4}{b(abc + a + c)} = \frac{4}{a + \frac{c}{bc+1}} : \frac{b}{ab+1} - \frac{4}{b(abc + a + c)} =$$

$$\frac{4}{\frac{abc+a+c}{bc+1}} \cdot \frac{ab+1}{b} - \frac{4}{b(abc+a+c)} = \frac{4(bc+1)}{abc+a+c} \cdot \frac{ab+1}{b} - \frac{4}{b(abc+a+c)} =$$

$$\frac{4(bc+1)(ab+1)}{b(abc+a+c)} - \frac{4}{b(abc+a+c)} = \frac{4[(bc+1)(ab+1)-1]}{b(abc+a+c)} = \frac{4[ab^2c+bc+ab+1-1]}{b(abc+a+c)} = \frac{4b(abc+c+a)}{b(abc+c+a)} = 4$$

11) IF  $a+b+c=0$  then  $a^3+b^3+c^3=3abc$ . Prove this!

$$a+b+c=0$$

$$a+b=-c$$

$$(a+b)^3 = (-c)^3$$

$$a^3 + \underline{3a^2b} + \underline{3ab^2} + b^3 = -c^3$$

$$a^3 + b^3 + 3ab(a+b) = -c^3 \rightarrow a+b = -c$$

$$a^3 + b^3 - 3abc = -c^3$$

$$a^3 + b^3 + c^3 = 3abc$$

12) If  $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$  then  $\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} = -3$ . Proof !

$$\frac{1}{a} + \frac{1}{b} = -\frac{1}{c}$$

$$\frac{b+a}{ab} = -\frac{1}{c}$$

$$a+b = -\frac{ab}{c} \quad / \text{ Share with c to create a phrase from the task}$$

$$\frac{a+b}{c} = -\frac{ab}{c^2}$$

$$\frac{b+c}{a} = -\frac{bc}{a^2}$$

$$\frac{c+a}{b} = -\frac{ca}{b^2}$$

$$\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} = -\frac{bc}{a^2} - \frac{ac}{b^2} - \frac{ab}{c^2} = -\frac{abc}{a^3} - \frac{abc}{b^3} - \frac{abc}{c^3} = -abc \left( \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} \right)$$

$$\frac{1}{a} + \frac{1}{b} = -\frac{1}{c} \quad / ()^3$$

$$\frac{1}{a^3} + 3 \cdot \frac{1}{a^2} \cdot \frac{1}{b} + 3 \cdot \frac{1}{a} \cdot \frac{1}{b^2} + \frac{1}{b^3} = -\frac{1}{c^3}$$

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{3}{ab} \left( \frac{1}{a} + \frac{1}{b} \right) = -\frac{1}{c^3}$$

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{3}{ab} \cdot \left( -\frac{1}{c} \right) = -\frac{1}{c^3}$$

$$\frac{1}{a^3} + \frac{1}{b^3} - \frac{3}{abc} = -\frac{1}{c^3}$$

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} = +\frac{3}{abc}$$

$$-abc \left( \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} \right) =$$

$$= -abc \cdot \frac{3}{abc} = -3$$