

LOGIC

Statement

In everyday speech, in written text, we have sentences that are true or false, sentences that are logical correct.

Those sentences in mathematic we call statements. **The statement is any sentence you knows that it can only be correct or only incorrect.** In other words, a statement can only have one of the two values: true (correct), false (incorrect).

Example:

It is not difficult to see which of the following sentence are statements:

- Number 6 is greater than the number 2
- Number 3 is divisible by number 2
- Earth revolves around the Sun
- Issue 2 is greater than Natasa
- Year has 365 days

The first three sentences are statements, because they are correct or incorrect, while the last two sentence we can not say, therefore, they are not statements.

Statements will, by agreement, note with : p, q, r, s, t. ...

From such fundamental statements we create complex statements.

Will be important for us to know when new statements will be correct or incorrect.

We will use:

T – for correct and \perp -for incorrect

$\tau(t)=T$, if a statement t is correct

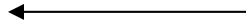
$\tau(t)=\perp$, if a statement t is incorrect

LOGICAL OPERATIONS

Conjunction (and) of statement p and q is statement $p \wedge q$ which corresponds to the following table:

p	q	$p \wedge q$
T	T	T
T	\perp	\perp
\perp	T	\perp
\perp	\perp	\perp

It is true if and only if both of its operands are true.



Disjunction (or) of statement p and q is statement $p \vee q$:

p	q	$p \vee q$
T	T	T
T	\perp	T
\perp	T	T
\perp	\perp	\perp

It is false only if both of its operands are false.



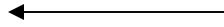
You need to pay attention to the difference between disjunction and exclusive disjunction which corresponds to the language form "**or .. or**".

The difference is that the statement "or p or q " is not correct in the case when both statements are true, while the statement " p or q " in this case is correct.

Implication (if...then) of statement p and q is statement $p \Rightarrow q$:

p	q	$p \Rightarrow q$
T	T	T
T	\perp	\perp
\perp	T	T
\perp	\perp	T

It is false just in case the first operand is true and the second operand is false.



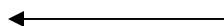
One of the most important “payer” for us is implication. Sentence **p implies q** with unchanged meaning can be record at one of the following ways:

- “If p, then q”
- “From p follows q “
- “q, if p”
- “p is a sufficient condition for q ”
- “q is necessary condition for the p”

Equivalence (if and only if) is statement $p \Leftrightarrow q$:

p	q	$p \Leftrightarrow q$
T	T	T
T	\perp	\perp
\perp	T	\perp
\perp	\perp	T

equivalence is correct only if both statements are true or both statements are false.



Sentence “**p is equivalent to q** “ we can say at one of the following ways:

- “If p, then q, and if q, then p”
- “p is necessary and sufficient condition for q ”
- “ p if and only if q”

These were the basic **binary** logic operations. Binary, because the two statements made a new statement.

Unary operation is \neg (not).

Negation of statement p is statement $\neg p$ which corresponds to table:

P	$\neg p$
T	\perp
\perp	T

Negation is operation of the highest priority! Then conjunction and disjunction, which are each equal, at the end are implication and equivalence, also mutually equal.

Example:

A series of symbol $p \wedge q \vee r$ **we can not accept** as a formula, because we do not know the order of operations!

This should be written in the following manner: $(p \wedge q) \vee r$ or $p \wedge (q \vee r)$.

Therefore, the statements of the representing formula established by letters p, q, r, \dots , characters $\wedge, \vee, \Rightarrow, \Leftrightarrow, \neg$, and brackets, using the final number of times these symbols.

Formulas that are true, for all possible values of letters that make the formula, we called a tautology.

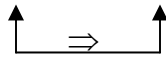
Whether the formula is tautology we can check in several ways: the discussion by letter, decrease the discrepancy, over the table of value, etc..

Here's one example of testing over table:

$$F: (\neg q \Rightarrow \neg p) \Rightarrow (p \Rightarrow q)$$

Take heed: Negation is “oldest operations”

p	q	$\neg q$	$\neg p$	$\neg q \Rightarrow \neg p$	$p \Rightarrow q$	$(\neg q \Rightarrow \neg p) \Rightarrow (p \Rightarrow q)$
T	T	⊥	⊥	T	T	T
T	⊥	T	⊥	⊥	⊥	T
⊥	T	⊥	T	T	T	T
⊥	⊥	T	T	T	T	T



Some rules of logical conclusion

$$p \wedge (p \Rightarrow q) \Rightarrow q \quad \text{modus ponens}$$

$$(\neg q \Rightarrow \neg p) \Rightarrow (p \Rightarrow q) \quad \text{contraposition rule (examined in the table)}$$

$$(\neg p \Rightarrow (q \wedge \neg q)) \Rightarrow p \quad \text{decrease in contradiction}$$

$$p \vee \neg p \quad \text{law out of the third}$$

$$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q \quad \text{De Morgan laws}$$

$$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$$

$$\neg\neg p \Leftrightarrow p \quad \text{dual negation law, etc..}$$

modus ponens by the use of the contradiction:

$$p \wedge (p \Rightarrow q) \Rightarrow q$$

Suppose that the formula is incorrect. Since the implications is incorrect only in one case, it must be:

$$\tau(p \wedge (p \Rightarrow q)) = T \quad \text{and} \quad \tau(q) = \perp$$

Further, as conjunction is correct only if both terms are correct, it must be:

$$\tau(p) = T \quad \text{and} \quad \tau(p \Rightarrow q) = T$$

Let we see table for implication:

p	q	$p \Rightarrow q$
T	T	T
T	\perp	\perp
\perp	T	T
\perp	\perp	T

As we have concluded that : $\tau(p) = T$ and $\tau(q) = \perp$ it must be $\tau(p \Rightarrow q) = \perp$,

which is in contradiction with $\tau(p \Rightarrow q) = T$.

This means that starting assumption is not good, and that the formula is tautology (correct).

Example:

Examine whether the following formula tautology:

$$F: (\neg p \Rightarrow q) \Leftrightarrow ((r \vee p) \wedge q)$$

When we have three of the representing letters, we need 8 types:

p	q	r	$\neg p$	$\neg p \Rightarrow q$	$r \vee p$	$(r \vee p) \wedge q$	F
T	T	T	⊥	T	T	T	T
T	T	⊥	⊥	T	T	T	T
T	⊥	T	⊥	T	T	⊥	⊥
T	⊥	⊥	⊥	T	⊥	⊥	⊥
⊥	T	T	T	T	T	T	T
⊥	T	⊥	T	T	T	T	T
⊥	⊥	T	T	⊥	T	⊥	T
⊥	⊥	⊥	T	⊥	⊥	⊥	T

So this formula is not tautology, because all values are not true. 

Quantification

Look the sentence: “ $x^2 = 25$ “.Obviously, it is **not** a statement, as it may be correct if $x = 5$ or $x = -5$, and may be invalid if x is some other number.

If, however, we say a sentence:

“For each x , $x^2 = 25$ ” or “There is x , $x^2 = 25$ ” then for them, we can say that first is incorrect and the second is correct, and **they are statements**.

Using mathematical terminology we can write:

“For each x , $x^2 = 25$ ” \longrightarrow $(\forall x) (x^2 = 25)$

“There is x , $x^2 = 25$ ” \longrightarrow $(\exists x) (x^2 = 25)$

We use the universal quantifier symbol \forall (an upside – down letter “A”) to indicate **universal quantification**.

\forall - we read : “for all”; “for each”

\exists - we read : 'there exists', and called **existential** quantification.

It is interesting how quantifiers **act** in **the presence of negation**:

$$\neg(\forall x)A \Leftrightarrow (\exists x)\neg A \quad \text{and}$$

$$\neg(\exists x)A \Leftrightarrow (\forall x)\neg A$$

For example, the sentence: "Not every professor is good" and "There is the professor who is not good" have the same meaning.