Inverse function

If f is a function from A to B then an **inverse function** for f is a function in the opposite direction, from B to A.

f: A $\rightarrow$ B

Not all functions have an inverse. Two conditions are:

1) Every $y \in B$ corresponds to **no more than one** $x \in A$; a function f with this property is called "one-to-one", or information-preserving, or an **injection.**

$$(\forall x_1, x_2 \in A)(x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2))$$

2) Every $y \in B$ corresponds to **at least one** $x \in A$; a function f with this property is called onto, or a surjection.

$$(\forall y \in B)(\exists x \in A)(f(x) = y)$$

The procedure for solving problems:

i) Instead of f (x) place y

ii) From here express x "over y"

iii) Process change: instead of x write f^{-1} (x) , and instead of y write x

EXAMPLES:

1) We have function f(x) = 2x – 1. Determine its inverse function and create a graphics function f(x) and

f^{-1}(x).

Solution:

f(x) = 2x – 1 ***Instead of f (x) place y***

y = 2x – 1 ***From here express x "over y"***

2x = y + 1

x = $\frac{y+1}{2}$ ***instead of x write f^{-1} (x) , and instead of y write x***

f^{-1}(x) = $\frac{x+1}{2}$ and here we inverse function.

Create a graphics function f(x) and $f^{-1}(x)$:

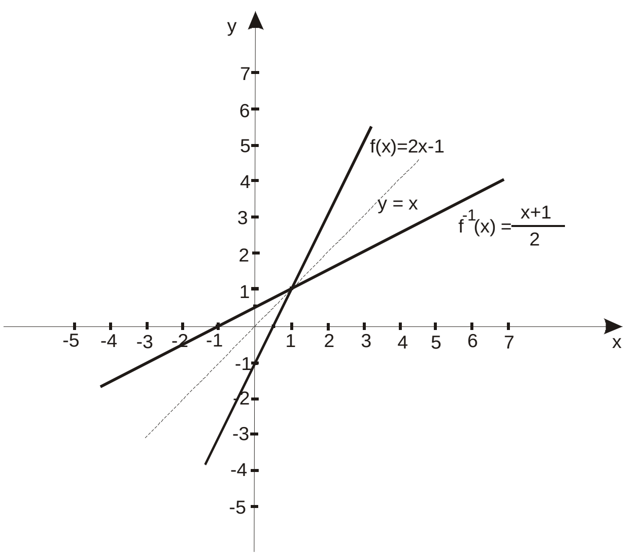
We'll take two arbitrary points (first x = 0 and then y = 0) and draw them.

f(x) = 2x – 1

x	0	1/2
f(x)	-1	0

$f^{-1}(x) = \frac{x+1}{2}$

x	0	-1
$f^{-1}(x)$	1/2	0



Note that graphics are balanced in relation to the y = x.

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2. We have function: $f(x) = \log_2(x-1)$. Determine its inverse function and create a graphics function f(x) and $f^{-1}(x)$.

Solution:

$f(x) = \log_2(x-1)$ ***Instead of f (x) place y***

$y = \log_2 (x-1)$ ***From here express x***

$x - 1 = 2^y$

$x = 2^y + 1$ ***instead of x write $f^{-1}(x)$, and instead of y write x***

$f^{-1}(x) = 2^x + 1$ and here is inverse function

graphics:

$f(x) = \log_2(x-1)$

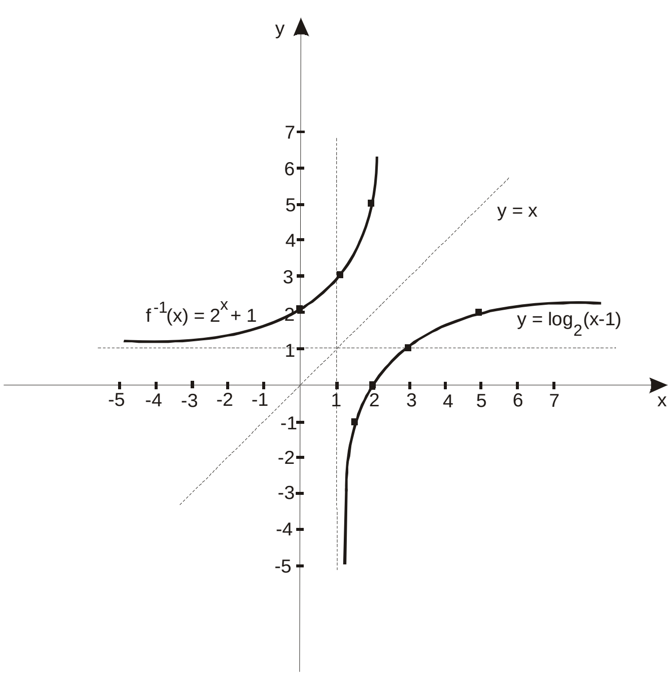
This function is defined for $x-1 > 0$, $x > 1$, which tells us that $x = 1$ is vertical asymptote on the left side. Take some arbitrary value and fill out the table:

x	3/2	2	3	5
f(x)	-1	0	1	2

$f^{-1}(x) = 2^x + 1$

This function obviously can not have a value of less than, or equal to 1, which tells us that 1 is its horizontal asymptote. Take some arbitrary value and fill out the table:

x	- 1	0	1
$f^{-1}(x)$	3/2	2	3



Note, again, that graphics are balanced in relation to the y = x

3) Determine the inverse function of function: $f(x) = 3^x - 1$

Solution:

$f(x) = 3^x - 1$

$y = 3^x - 1$

$3^x = y + 1$

$x = \log_3(y + 1)$

$f^{-1}(x) = \log_3(x + 1)$

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4) We have function: $f(x) = x^2$. Determine its inverse function $f^{-1}(x)$.

Solution:

$f(x) = x^2$

$y = x^2$

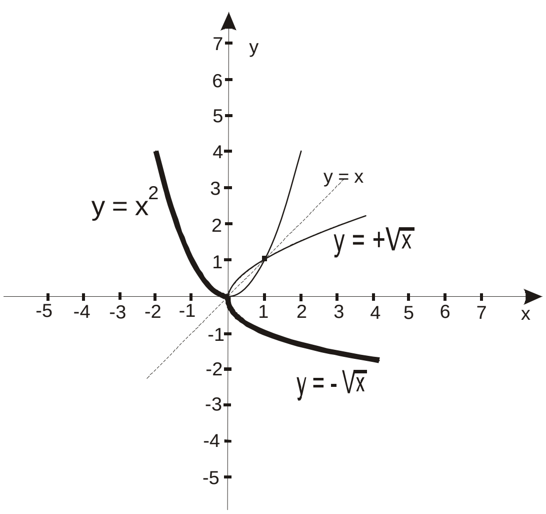
$x = \pm\sqrt{y} \longrightarrow f^{-1}(x) = \pm\sqrt{x}$

It was not difficult to solve this, but this solution is not fair! Why?

Must take into account where the function is increasing, and where decreasing!

$f(x) = x^2$ is decreasing for $x < 0$ and for her is : $f^{-1}(x) = -\sqrt{x}$
$f(x) = x^2$ is increasing for $x > 0$ and for her is : $f^{-1}(x) = +\sqrt{x}$

This is the correct solution now!



5) We have function: f(x) = $\log_2(x + \sqrt{x^2+1})$. Find $f^{-1}(x)$.

Solution:

$f(x) = \log_2(x + \sqrt{x^2+1})$

$y = \log_2(x + \sqrt{x^2+1})$

$x + \sqrt{x^2+1} = 2^y$

$\sqrt{x^2+1} = 2^y - x$

$x^2 + 1 = 2^{2y} - 2x\, 2^y + x^2$

$2x\, 2^y = 2^{2y} - 1$

$x = \dfrac{2^{2y}-1}{2^{y+1}}$

$f^{-1}(x) = \dfrac{2^{2x}-1}{2^{x+1}}$

$f^{-1}(x) = \dfrac{2^{2x}-1}{2^{x+1}} = \dfrac{2^{2x}-1}{2^x 2} = \dfrac{\frac{2^{2x}}{2^x} - \frac{1}{2^x}}{2} = \dfrac{2^x - 2^{-x}}{2}$

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6) We have function: $f(x) = \sqrt[3]{x+\sqrt{1+x^2}} + \sqrt[3]{x-\sqrt{1+x^2}}$. Find $f^{-1}(x)$.

Solution:

$f(x) = \sqrt[3]{x+\sqrt{1+x^2}} + \sqrt[3]{x-\sqrt{1+x^2}}$

$y = \sqrt[3]{x+\sqrt{1+x^2}} + \sqrt[3]{x-\sqrt{1+x^2}}$ This all goes to the third degree.

Remind yourselfe formula:

$(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3 = A^3 + 3AB(A+B) + B^3$

$y^3 = x+\sqrt{1+x^2} + 3\sqrt[3]{x+\sqrt{1+x^2}}\ \sqrt[3]{x-\sqrt{1+x^2}}\ (\sqrt[3]{x+\sqrt{1+x^2}} + \sqrt[3]{x-\sqrt{1+x^2}}) + x-\sqrt{1+x^2}$

$y^3 = 2x + 3\sqrt[3]{(x+\sqrt{1+x^2})(x-\sqrt{1+x^2})}\ y$

$y^3 = 2x + 3\sqrt[3]{x^2 - 1 - x^2}\ y$

$y^3 = 2x - 3y$

$2x = y^3 + 3y$

$$x = \frac{y^3 + 3y}{2}$$

$$f^{-1}(x) = \frac{x^3 + 3x}{2}$$ **final solution**