

Functional equation

Solving process:

- i) "It", which is in brackets replace with t (replacement)
- ii) From there, express x
- iii) Back in the starting equation, f (t) =... and where you see x replace it with what we have expressed
- iv) Simplify that term, who is now all "by t" and **replace t with x**

EXAMPLES:

1) **Solve the functional equation:** $f(x+1) = x^2 - 3x + 2$

Solution:

$$f(x+1) = x^2 - 3x + 2 \quad \text{"It", which is in brackets replace with t}$$

$$x + 1 = t \quad \text{From there, express x}$$

$$x = t - 1 \quad \text{Back in the starting equation, f (t) =...}$$

$$f(t) = (t-1)^2 - 3(t-1) + 2$$

$$f(t) = t^2 - 2t + 1 - 3t + 3 + 2 \quad \text{Simplify that term, who is now all "by t"}$$

$$f(t) = t^2 - 5t + 6 \quad \text{replace t with x}$$

$$f(x) = x^2 - 5x + 6 \longrightarrow \text{final solution of the functional equation.}$$

2) **Solve the functional equation:** $f\left(\frac{1}{x}\right) = x + \sqrt{1+x^2}$

Solution:

$$f\left(\frac{1}{x}\right) = x + \sqrt{1+x^2}$$

$$\frac{1}{x} = t$$

$$f(t) = \frac{1}{t} + \sqrt{1 + \frac{1}{t^2}}$$

$$f(t) = \frac{1}{t} + \sqrt{\frac{t^2+1}{t^2}} \longrightarrow f(t) = \frac{1}{t} + \frac{\sqrt{t^2+1}}{t} \quad \text{replace t with x} \quad f(x) = \frac{1 + \sqrt{x^2+1}}{x} \text{ is final solution}$$

3) Solve the functional equation: $f\left(\frac{x}{x+1}\right) = x^2$

Solution:

$$f\left(\frac{x}{x+1}\right) = x^2$$

$$\frac{x}{x+1} = t$$

$$x = t(x+1)$$

$$x = tx + t$$

$$x - tx = t$$

$$x(1-t) = t$$

$$x = \frac{t}{1-t}$$

$$f\left(\frac{x}{x+1}\right) = x^2$$

$$f(t) = \left(\frac{t}{1-t}\right)^2 \quad \text{replace } t \text{ with } x \quad f(x) = \left(\frac{x}{1-x}\right)^2 \text{ is final solution}$$

4) Solve the functional equation: $f\left(\frac{x+2}{2x+1}\right) = 5x+3$

Solution:

$$f\left(\frac{x+2}{2x+1}\right) = 5x+3$$

$$\frac{x+2}{2x+1} = t$$

$$x+2 = t(2x+1)$$

$$x+2 = 2tx+t$$

$$x-2tx = t-2$$

$$x(1-2t) = t-2$$

$$x = \frac{t-2}{1-2t} \quad \longrightarrow \quad f\left(\frac{x+2}{2x+1}\right) = 5x+3$$

$$f(t) = 5 \frac{t-2}{1-2t} + 3$$

$$f(t) = \frac{5t-10}{1-2t} + \frac{3(1-2t)}{1-2t} = \frac{5t-10+3-6t}{1-2t} = \frac{-t-7}{1-2t}$$

$$f(t) = \frac{t+7}{2t-1}$$

$$f(x) = \frac{x+7}{2x-1} \quad \text{is final solution.}$$

5) If $f\left(\frac{x}{x+1}\right) = (x-1)^2$ calculate $f(3)$.

Solution:

First, we must find $f(x)$.

$$f\left(\frac{x}{x+1}\right) = (x-1)^2$$

$$\frac{x}{x+1} = t$$

$$x = t(x+1)$$

$$x = tx + t$$

$$x - tx = t$$

$$x(1-t) = t$$

$$x = \frac{t}{1-t} \quad \longrightarrow \quad f\left(\frac{x}{x+1}\right) = (x-1)^2$$

$$f(t) = \left(\frac{t}{1-t} - 1\right)^2 \quad \text{Now, } t \text{ replace with } 3, \text{ because we search } f(3) \dots$$

$$f(3) = \left(\frac{3}{1-3} - 1\right)^2 = \frac{25}{4}$$

6) Solve the functional equation: $f\left(x + \frac{1}{x}\right) = x^2 + \frac{1}{x^2}$

Solution:

$f\left(x + \frac{1}{x}\right) = x^2 + \frac{1}{x^2}$ replacement $x + \frac{1}{x} = t$, if you try from here to express x , as it should, **there is a problem....**

$x + \frac{1}{x} = t$ all multiply with x ..
 $x^2 + 1 = xt$

$x^2 - xt + 1 = 0$ this is a square equation but does not give us a “good” solution

What should you do?

Must use a different method:

$x + \frac{1}{x} = t$

$\left(x + \frac{1}{x}\right)^2 = t^2$

$x^2 + 2x \frac{1}{x} + \frac{1}{x^2} = t^2$

$x^2 + 2 + \frac{1}{x^2} = t^2$

$x^2 + \frac{1}{x^2} = t^2 - 2$ now back in the equation $f\left(x + \frac{1}{x}\right) = x^2 + \frac{1}{x^2}$

$f\left(x + \frac{1}{x}\right) = x^2 + \frac{1}{x^2} \longrightarrow f(t) = t^2 - 2$ then $f(x) = x^2 - 2$ is final solution

7) Solve the functional equation: $f\left(\frac{x+1}{x-2}\right) + 2f\left(\frac{x-2}{x+1}\right) = x$

Solution:

This task can not do "classic" but must use a different method:

If we have replacement $\frac{x-2}{x+1} = t$ then $\frac{x+1}{x-2} = \frac{1}{t}$ and from here we have:

$$x-2 = t(x+1) \longrightarrow x-2 = tx+t \longrightarrow x-tx = t+2 \longrightarrow x(1-t) = t+2 \longrightarrow x = \frac{t+2}{1-t}$$

Back in the equation:

$$f\left(\frac{x+1}{x-2}\right) + 2f\left(\frac{x-2}{x+1}\right) = x$$

$$f\left(\frac{1}{t}\right) + 2f(t) = \frac{t+2}{1-t} \quad \text{We have received new equation, trick is to place } \frac{1}{t} \text{ instead of } t.$$

$$f(t) + 2f\left(\frac{1}{t}\right) = \frac{\frac{1}{t} + 2}{1 - \frac{1}{t}} = \frac{\frac{1+2t}{t}}{\frac{t-1}{t}} = \frac{1+2t}{t-1}$$

Now make the system of this two equations:

$$\begin{aligned} f\left(\frac{1}{t}\right) + 2f(t) &= \frac{t+2}{1-t} \\ 2f\left(\frac{1}{t}\right) + f(t) &= \frac{1+2t}{t-1} \end{aligned}$$

Multiply the first equation with -2 and gather these two equations ...

$$\begin{aligned} -4f(t) - 2f\left(\frac{1}{t}\right) &= -2 \frac{t+2}{1-t} \\ f(t) + 2f\left(\frac{1}{t}\right) &= \frac{1+2t}{t-1} \end{aligned}$$

$$-3f(t) = \frac{-2t-4}{1-t} + \frac{1+2t}{t-1} = \frac{2t+4}{t-1} + \frac{1+2t}{t-1} = \frac{4t+5}{t-1}$$

$$\text{So: } -3f(t) = \frac{4t+5}{t-1}$$

$$f(t) = \frac{4t+5}{-3(t-1)} \longrightarrow f(t) = \frac{4t+5}{3-3t} \quad \text{replace } t \text{ with } x$$

$$f(x) = \frac{4x+5}{3-3x} \quad \text{is final solution}$$

