Extremes of functions with multiple variables (part II)

conditional extremes function

Here we have set apart the functions and conditions. Most often it is **one condition ,** but in more serious instances could be two or more.

If we have function f (x) and condition φ (x) , we first set up a function:

$$F(x) = f(x) + \lambda \cdot \varphi(x)$$

λ is an unknown coefficient that we ask:

- Find the first partial derivatives and equate them to zero
- Express x and y over λ
- Replace x and y in condition φ (x) and find the value for λ (There may be several)
- Return that value in x and y , so we have stationary points
- Look for second total differential d^2F to examine whether it comes to maximum or minimum (If $d^2F < 0$ then it is max , and if $d^2F > 0$, then it is min.)

Example 1.

Find conditional extremes for function $z = ax + by$ **if the condition is** $x^2 + y^2 = 1$

Solution:

So, first set up a function:

$F = ax + by + \lambda(x^2 + y^2 - 1)$ watch out, placed a condition equal to zero! $x^2 + y^2 = 1 \rightarrow \boxed{x^2 + y^2 - 1 = 0}$

Still looking for first partial derivatives and equate them with 0.

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$$F = ax + by + \lambda(x^2 + y^2 - 1)$$

$$\frac{\partial F}{\partial x} = a + 2\lambda x$$

$$\frac{\partial F}{\partial y} = a + 2\lambda y$$

$$a + 2\lambda x = 0 \rightarrow \boxed{x = -\frac{a}{2\lambda}}$$

$$b + 2\lambda y = 0 \rightarrow \boxed{y = -\frac{b}{2\lambda}}$$

These replace in condition to find a value for λ:

$$x^2 + y^2 = 1$$

$$(-\frac{a}{2\lambda})^2 + (-\frac{b}{2\lambda})^2 = 1$$

$$\frac{a^2}{4\lambda^2} + \frac{b^2}{4\lambda^2} = 1$$

$$a^2 + b^2 = 4\lambda^2 \rightarrow \lambda^2 = \frac{a^2 + b^2}{4} \rightarrow \lambda = \pm\sqrt{\frac{a^2 + b^2}{4}} \rightarrow \lambda = \pm\frac{\sqrt{a^2 + b^2}}{2}$$

$$\boxed{\lambda_1 = +\frac{\sqrt{a^2 + b^2}}{2}} \quad \wedge \quad \boxed{\lambda_2 = -\frac{\sqrt{a^2 + b^2}}{2}}$$

We have two values, which means that we have two stationary points. First, we replace λ_1:

$$\lambda_1 = +\frac{\sqrt{a^2 + b^2}}{2}$$

$$x = -\frac{a}{2\lambda} \rightarrow x = -\frac{a}{\cancel{2}\frac{\sqrt{a^2 + b^2}}{\cancel{2}}} \rightarrow \boxed{x = -\frac{a}{\sqrt{a^2 + b^2}}}$$

$$y = -\frac{b}{2\lambda} \rightarrow y = -\frac{b}{\cancel{2}\frac{\sqrt{a^2 + b^2}}{\cancel{2}}} \rightarrow \boxed{y = -\frac{b}{\sqrt{a^2 + b^2}}}$$

$$\boxed{M(-\frac{a}{\sqrt{a^2 + b^2}}, -\frac{b}{\sqrt{a^2 + b^2}})}$$

Investigate whether the maximum or minimum over the second-order total differential:

$$\frac{\partial F}{\partial x} = a + 2\lambda x$$
$$\frac{\partial F}{\partial y} = a + 2\lambda y$$

$$\frac{\partial^2 F}{\partial x^2} = 2\lambda$$
$$\frac{\partial^2 F}{\partial y^2} = 2\lambda$$
$$\frac{\partial^2 F}{\partial x \partial y} = 0$$

from here we have :

$$d^2F = \frac{\partial^2 F}{\partial x^2} dx^2 + 2\frac{\partial^2 F}{\partial x \partial y} dxdy + \frac{\partial^2 F}{\partial y^2} dy^2$$
$$d^2F = 2\lambda dx^2 + 0 + 2\lambda dy^2$$
$$\boxed{d^2F = 2\lambda(dx^2 + dy^2)}$$
$$\lambda_1 = +\frac{\sqrt{a^2+b^2}}{2}$$
$$d^2F = \cancel{2}\frac{\sqrt{a^2+b^2}}{\cancel{2}}(dx^2 + dy^2)$$
$$d^2F = \sqrt{a^2+b^2} \cdot (dx^2 + dy^2) \rightarrow \boxed{d^2F > 0}$$

As is $d^2F > 0$, point is **minimum**, replace in the initial function to find the minimum value:

$$z_{\min} = ax + by$$
$$z_{\min} = a(-\frac{a}{\sqrt{a^2+b^2}}) + b(-\frac{b}{\sqrt{a^2+b^2}})$$
$$z_{\min} = -\frac{a^2}{\sqrt{a^2+b^2}} - \frac{b^2}{\sqrt{a^2+b^2}}$$
$$z_{\min} = -\frac{a^2+b^2}{\sqrt{a^2+b^2}}$$
$$\boxed{z_{\min} = -\sqrt{a^2+b^2}}$$

Now do the same procedure for another value $\lambda_2 = -\frac{\sqrt{a^2+b^2}}{2}$:

$$\lambda_2 = -\frac{\sqrt{a^2+b^2}}{2}$$
$$x = -\frac{a}{2\lambda} \rightarrow x = -\frac{a}{-\cancel{2}\frac{\sqrt{a^2+b^2}}{\cancel{2}}} \rightarrow \boxed{x = \frac{a}{\sqrt{a^2+b^2}}}$$
$$y = -\frac{b}{2\lambda} \rightarrow y = -\frac{b}{-\cancel{2}\frac{\sqrt{a^2+b^2}}{\cancel{2}}} \rightarrow \boxed{y = \frac{b}{\sqrt{a^2+b^2}}}$$

$$\boxed{N(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}})}$$

$$\boxed{d^2F = 2\lambda(dx^2 + dy^2)}$$

$$\lambda_2 = -\frac{\sqrt{a^2+b^2}}{2}$$

$$d^2F = -\cancel{2}\frac{\sqrt{a^2+b^2}}{\cancel{2}}(dx^2 + dy^2)$$

$$d^2F = -\sqrt{a^2+b^2}\cdot(dx^2 + dy^2) \to \boxed{d^2F < 0}$$

We conclude that this is the point of **maximum,** and to determine the maximum function value, we do:

$$z_{\max} = ax + by$$

$$z_{\max} = a\frac{a}{\sqrt{a^2+b^2}} + b\frac{b}{\sqrt{a^2+b^2}}$$

$$z_{\max} = \frac{a^2}{\sqrt{a^2+b^2}} + \frac{b^2}{\sqrt{a^2+b^2}}$$

$$z_{\max} = \frac{a^2+b^2}{\sqrt{a^2+b^2}}$$

$$\boxed{z_{\max} = \sqrt{a^2+b^2}}$$

Example 2.

Find conditional extremes for function $u = x - 2y + 2z$ **if the condition is** $x^2 + y^2 + z^2 = 1$

Solution:

$$F = x - 2y + 2z + \lambda(x^2 + y^2 + z^2 - 1)$$

$$F = x - 2y + 2z + \lambda(x^2 + y^2 + z^2 - 1)$$

$$\frac{\partial F}{\partial x} = 1 + 2\lambda x \qquad 1 + 2\lambda x = 0 \to x = -\frac{1}{2\lambda}$$

$$\frac{\partial F}{\partial y} = -2 + 2\lambda y \qquad -2 + 2\lambda y = 0 \to y = \frac{1}{\lambda}$$

$$\frac{\partial F}{\partial z} = 2 + 2\lambda z \qquad 2 + 2\lambda z = 0 \to z = -\frac{1}{\lambda}$$

$x^2+y^2+z^2=1$

$(-\frac{1}{2\lambda})^2+(\frac{1}{\lambda})^2+(-\frac{1}{\lambda})^2=1$

$\frac{1}{4\lambda^2}+\frac{1}{\lambda^2}+\frac{1}{\lambda^2}=1 .... / \cdot 4\lambda^2$

$1+4+4=4\lambda^2$

$\lambda^2=\frac{9}{4}\rightarrow\lambda=\pm\frac{3}{2}\rightarrow\boxed{\lambda_1=+\frac{3}{2}}\vee\boxed{\lambda_2=-\frac{3}{2}}$

Again we have **two values for** λ **:**

For $\lambda_1=+\frac{3}{2}$

$\frac{\partial F}{\partial x}=1+2\lambda x\rightarrow\frac{\partial^2 F}{\partial x^2}=2\lambda$

$\frac{\partial F}{\partial y}=-2+2\lambda y\rightarrow\frac{\partial^2 F}{\partial y^2}=2\lambda$

$\frac{\partial F}{\partial z}=2+2\lambda z\rightarrow\frac{\partial^2 F}{\partial z^2}=2\lambda$

$d^2F=2\lambda d^2x+2\lambda d^2y+2\lambda d^2z=2\lambda(d^2x+d^2y+d^2z)$

$d^2F=\cancel{2}\cdot\frac{3}{\cancel{2}}(d^2x+d^2y+d^2z)$

$d^2F=3(d^2x+d^2y+d^2z)\rightarrow\boxed{d^2F>0}$

$M(-\frac{1}{3},\frac{2}{3},-\frac{2}{3})$ is minimum

$u=x-2y+2z$

$u_{(-\frac{1}{3},\frac{2}{3},-\frac{2}{3})}=-\frac{1}{3}-2\cdot\frac{2}{3}+2\cdot(-\frac{2}{3})=-3$

$\boxed{u_{\min}=-3}$

For $\lambda_2 = -\frac{3}{2}$

$$\frac{\partial F}{\partial x} = 1 + 2\lambda x \to \frac{\partial^2 F}{\partial x^2} = 2\lambda$$
$$\frac{\partial F}{\partial y} = -2 + 2\lambda y \to \frac{\partial^2 F}{\partial y^2} = 2\lambda$$
$$\frac{\partial F}{\partial z} = 2 + 2\lambda z \to \frac{\partial^2 F}{\partial z^2} = 2\lambda$$

$$x = -\frac{1}{2\lambda} \to \boxed{x = \frac{1}{3}}$$
$$y = \frac{1}{\lambda} \to \boxed{y = -\frac{2}{3}}$$
$$z = -\frac{1}{\lambda} \to \boxed{z = +\frac{2}{3}}$$
$$\boxed{N(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3})}$$

then is

$$d^2F = 2\lambda d^2x + 2\lambda d^2y + 2\lambda d^2z = 2\lambda(d^2x + d^2y + d^2z)$$
$$d^2F = \cancel{2} \cdot (-\frac{3}{\cancel{2}})(d^2x + d^2y + d^2z)$$
$$d^2F = -3(d^2x + d^2y + d^2z) \to \boxed{d^2F < 0}$$

$N(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3})$ je maximum

$$u = x - 2y + 2z$$
$$u_{(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3})} = \frac{1}{3} - 2 \cdot (-\frac{2}{3}) + 2 \cdot \frac{2}{3} = 3$$
$$\boxed{u_{\max} = 3}$$

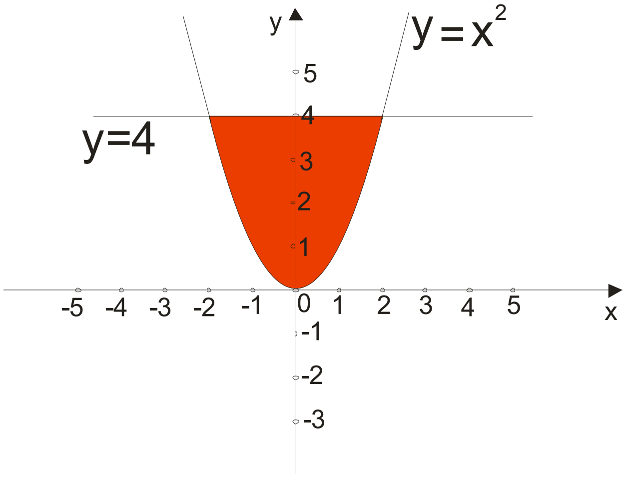
Differentiable function f (x) reaches its maximum or minimum value in a closed and limited area,in stationary point or point in the border area.

Example 3.

Find the minimum and maximum value for function $z = 2x^3 + 4x^2 + y^2 - 2xy$ **in a closed area limited with curves:** $y = x^2$ **and** $y = 4$

Solution:

Here we must first draw a picture and see the area:



Stationary point we are looking at the casual way:

$$z = 2x^3 + 4x^2 + y^2 - 2xy$$

$$\frac{\partial z}{\partial x} = 6x^2 + 8x - 2y$$

$$\frac{\partial z}{\partial y} = 2y - 2x$$

$$\frac{\partial z}{\partial x} = 0 \rightarrow 6x^2 + 8x - 2y = 0$$

$$\frac{\partial z}{\partial y} = 0 \rightarrow 2y - 2x = 0 \rightarrow \boxed{x = y}$$

$$x = y \quad \text{replace in} \quad \rightarrow 6x^2 + 8x - 2y = 0$$

$$6x^2 + 8x - 2x = 0 \rightarrow 6x^2 + 6x = 0 \rightarrow x_1 = 0, \;\; x_2 = -1$$

$$x_1 = 0 \rightarrow y_1 = 0 \rightarrow \boxed{M_1(0,0)}$$

$$x_2 = -1 \rightarrow \; \mathrm{y}_2 = -1 \rightarrow \boxed{M_2(-1,-1)}$$

Next we must determine the value of initial function in these points:

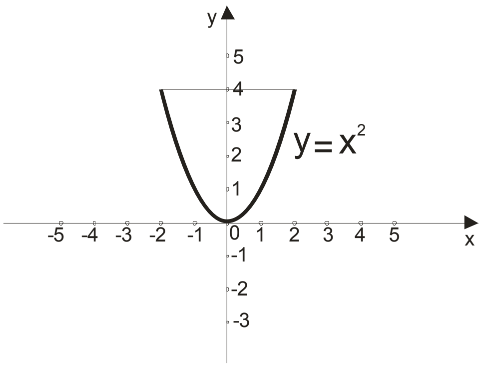
$$M_1(0,0)$$
$$z = 2x^3 + 4x^2 + y^2 - 2xy$$
$$z_{(0,0)} = 2 \cdot 0^3 + 4 \cdot 0^2 + 0^2 - 2 \cdot 0 \cdot 0 = 0$$
$$\boxed{z_{(0,0)} = 0}$$

and

$$M_2(-1,-1)$$
$$z = 2x^3 + 4x^2 + y^2 - 2xy$$
$$z_{(-1,-1)} = 2(-1)^3 + 4(-1)^2 + (-1)^2 - 2(-1)(-1) = -2 + 4 + 1 - 2 = 1$$
$$\boxed{z_{(-1,-1)} = 1}$$

For now **we do not know** whether this points are extreme points, we must examine border points.

First, we examine the **point of parable:**



Replace $y = x^2$ in a given function and looking derivate.

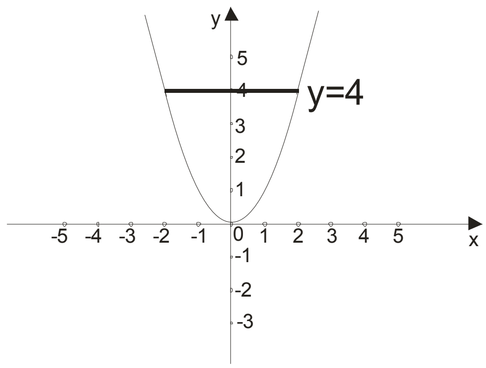
$$
\begin{aligned}
&y = x^2\\
&z = 2x^3 + 4x^2 + y^2 - 2xy\\
&z_{(y=x^2)} = 2x^3 + 4x^2 + (x^2)^2 - 2x(x^2)\\
&z_{(y=x^2)} = \cancel{2x^3} + 4x^2 + x^4 \cancel{-2x^3}\\
&z_{(y=x^2)} = x^4 + 4x^2\\
&derivate:\\
&z` = 4x^3 + 8x\\
&z` = 0 \rightarrow 4x^3 + 8x = 0 \rightarrow 4x(x^2 + 2) = 0 \rightarrow \boxed{x = 0}\\
&y = x^2 \rightarrow y = 0^2 \rightarrow \boxed{y = 0}
\end{aligned}
$$

We have point $\boxed{M_3(0,0)}$

This point we have already received as stationary …

$$
\begin{aligned}
&M_3(0,0) = M_1(0,0)\\
&z = 2x^3 + 4x^2 + y^2 - 2xy\\
&z_{(0,0)} = 2\cdot 0^3 + 4\cdot 0^2 + 0^2 - 2\cdot 0\cdot 0 = 0\\
&\boxed{z_{(0,0)} = 0}
\end{aligned}
$$

Now examine the point on the line ***y = 4***



$$y = 4$$
$$z = 2x^3 + 4x^2 + y^2 - 2xy$$
$$z_{(y=4)} = 2x^3 + 4x^2 + 4^2 - 2x \cdot 4$$
$$z_{(y=4)} = 2x^3 + 4x^2 - 8x + 16$$

derivate:

$$z` = 6x^2 + 8x - 8$$
$$z` = 0 \rightarrow 6x^2 + 8x - 8 = 0 \rightarrow x_1 = \frac{2}{3}, \quad x_2 = -2$$
$$\boxed{M_4 = (\frac{2}{3}, 4)}, \quad \boxed{M_5 = (-2, 4)}$$

$$M_4(\frac{2}{3}, 4)$$
$$z = 2x^3 + 4x^2 + y^2 - 2xy$$
$$z_{(\frac{2}{3},4)} = 2 \cdot (\frac{2}{3})^3 + 4 \cdot (\frac{2}{3})^2 + 4^2 - 2 \cdot (\frac{2}{3}) \cdot 4 = 0$$
$$z_{(\frac{2}{3},4)} = \frac{16}{27} + \frac{16}{9} + 16 - \frac{16}{3} = \frac{176}{27} = 6\frac{12}{27}$$
$$\boxed{z_{(\frac{2}{3},4)} = 6\frac{12}{27}}$$

and

$$M_5(-2, 4)$$
$$z = 2x^3 + 4x^2 + y^2 - 2xy$$
$$z_{(-2,4)} = 2 \cdot (-2)^3 + 4 \cdot (-2)^2 + 4^2 - 2 \cdot (-2) \cdot 4 = 0$$
$$z_{(-2,4)} = -16 + 16 + 16 + 16$$
$$\boxed{z_{(-2,4)} = 32}$$

So:

The function has a maximum value at point $M_5(-2, 4)$ **that is :** $z_{(-2,4)} = 32$

The function has a minimum value at point $M_3(0,0) = M_1(0,0)$ **that is :** $z_{(0,0)} = 0$