

## Extremes of functions with multiple variables (part I)

In most of these tasks is set of functions  $f(x, y)$ .

The first job is to find partial derivatives  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$ .

Then solve the system of equations  $\frac{\partial f}{\partial x} = 0$  and  $\frac{\partial f}{\partial y} = 0$ .

Solutions of this system (may be one, but several of them) give us a **stationary points**  $(x_0, y_0)$ ,  $(x_1, y_1)$ , ...etc.

Next we find  $A = \frac{\partial^2 f}{\partial x^2}$ ,  $B = \frac{\partial^2 f}{\partial x \partial y}$ ,  $C = \frac{\partial^2 f}{\partial y^2}$

Organize function  $D = A \cdot C - B^2$

**For each stationary point we doing the same thing:**

First, replace the value of **stationary points** in  $D$ . The resulting value must be greater than zero  $\boxed{D > 0}$ ,

If it occurs it is not, then that point is not the point of extreme.

Further investigate whether the maximum or minimum:

- i) If  $\boxed{D > 0}$  and  $A < 0$  ( $C < 0$ ) then the stationary point is **maximum**
- ii) If  $\boxed{D > 0}$  and  $A > 0$  ( $C > 0$ ) then the stationary point is **minimum**
- iii) If  $D = 0$  case is **indeterminate**
- iv) If  $D < 0$  then **no extreme**

**If it happens that  $D = 0$** , that is, that the case is vague, we have to go to a broader definition of extreme, that is,

looking for second-order differentia  $d^2 f$  and

- a) If  $d^2 f > 0$  function has a minimum
- b) If  $d^2 f < 0$  function has a maximum

**Example 1.**

**Find extreme for function**  $z = x^3 + 8y^3 - 6xy + 5$

**Solution:**

First, look for the first partial derivatives:

$$z = x^3 + 8y^3 - 6xy + 5$$

$$\frac{\partial z}{\partial x} = 3x^2 - 6y$$

$$\frac{\partial z}{\partial y} = 24y^2 - 6x$$

Establish a system of equations  $\frac{\partial z}{\partial x} = 0$  and  $\frac{\partial z}{\partial y} = 0$ .

$$3x^2 - 6y = 0 \dots / : 3$$

$$24y^2 - 6x = 0 \dots / : 6$$

$$x^2 - 2y = 0 \rightarrow y = \frac{x^2}{2} \quad \text{replace in } 4y^2 - x = 0$$

$$4y^2 - x = 0$$

$$4\left(\frac{x^2}{2}\right)^2 - x = 0 \rightarrow \cancel{4} \frac{x^4}{\cancel{4}} - x = 0 \rightarrow x^4 - x = 0 \rightarrow x(x^3 - 1) = 0$$

$$x = 0 \vee x = 1$$

$$\text{For } x = 0 \rightarrow y = \frac{x^2}{2} \rightarrow y = 0 \rightarrow \boxed{M_1(0,0)}$$

$$\text{For } x = 1 \rightarrow y = \frac{x^2}{2} \rightarrow y = \frac{1}{2} \rightarrow \boxed{M_1\left(1, \frac{1}{2}\right)}$$

We got two stationary points:  $M_1(0,0)$  and  $M_1\left(1, \frac{1}{2}\right)$ .

Next, we must find  $A = \frac{\partial^2 z}{\partial x^2}$ ,  $B = \frac{\partial^2 z}{\partial x \partial y}$ ,  $C = \frac{\partial^2 z}{\partial y^2}$  and make  $D = A \cdot C - B^2$

$$\frac{\partial z}{\partial x} = 3x^2 - 6y$$

$$\frac{\partial z}{\partial y} = 24y^2 - 6x$$

$$A = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(3x^2 - 6y) = 6x$$

$$B = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(24y^2 - 6x) = -6$$

$$C = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(24y^2 - 6x) = 48y$$

then is :

$$D = A \cdot C - B^2$$

$$D = (6x)(48y) - (-6)^2$$

$$D = 288xy - 36$$

Taking the first stationary point and investigate:

$$M_1(0, 0)$$

$$D = 288xy - 36 \rightarrow D(0, 0) = 288 \cdot 0 \cdot 0 - 36 = -36 \rightarrow D(0, 0) < 0$$

Therefore, we have shown that this point is not extreme!

Investigate the second stationary point:

$$M_1\left(1, \frac{1}{2}\right)$$

$$D = 288xy - 36 \rightarrow D\left(1, \frac{1}{2}\right) = 288 \cdot 1 \cdot \frac{1}{2} - 36 = 144 - 36 = 108 \rightarrow D\left(1, \frac{1}{2}\right) > 0$$

This point is extreme, even to investigate whether the max or min.

$$A = 6x \rightarrow A\left(1, \frac{1}{2}\right) = 6 \cdot 1 = 6 \rightarrow A\left(1, \frac{1}{2}\right) > 0$$

We conclude that this point is minimum!

This value back to the starting function to calculate the minimum value:

$$z = x^3 + 8y^3 - 6xy + 5$$

$$z_{\min}\left(1, \frac{1}{2}\right) = 1^3 + 8\left(\frac{1}{2}\right)^3 - 6 \cdot 1 \cdot \left(\frac{1}{2}\right) + 5 = 1 + 1 - 3 + 5 = 4$$

$$z_{\min}\left(1, \frac{1}{2}\right) = 4$$

**Example 2.**

**Find extreme for function :**  $z = x\sqrt{y} - x^2 - y + 6x + 3$

**Solution:**

$$z = x\sqrt{y} - x^2 - y + 6x + 3$$

$$\frac{\partial z}{\partial x} = \sqrt{y} - 2x + 6$$

$$\frac{\partial z}{\partial y} = \frac{x}{2\sqrt{y}} - 1$$

$$\sqrt{y} - 2x + 6 = 0$$

$$\frac{x}{2\sqrt{y}} - 1 = 0$$

Let  $\frac{x}{2\sqrt{y}} - 1 = 0 \rightarrow x = 2\sqrt{y}$ , then this replace in  $\sqrt{y} - 2x + 6 = 0$

$$\sqrt{y} - 2 \cdot 2\sqrt{y} + 6 = 0$$

$$-3\sqrt{y} + 6 = 0 \rightarrow \sqrt{y} = 2 \rightarrow \boxed{y = 4}$$

$$x = 2\sqrt{y} \rightarrow x = 2 \cdot \sqrt{4} \rightarrow \boxed{x = 4}$$

$\boxed{M(4,4)}$  is stationary point (only)

Still looking  $A = \frac{\partial^2 z}{\partial x^2}$ ,  $B = \frac{\partial^2 z}{\partial x \partial y}$ ,  $C = \frac{\partial^2 z}{\partial y^2}$  and  $D = A \cdot C - B^2$

$$\frac{\partial z}{\partial x} = \sqrt{y} - 2x + 6$$

$$\frac{\partial z}{\partial y} = \frac{x}{2\sqrt{y}} - 1$$

$$A = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(\sqrt{y} - 2x + 6) = -2$$

$$B = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{x}{2\sqrt{y}} - 1\right) = \frac{1}{2\sqrt{y}}$$

$$C = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{x}{2\sqrt{y}} - 1\right) = \frac{\partial}{\partial y}\left(\frac{x}{2} \cdot y^{-\frac{1}{2}} - 1\right) = -\frac{x}{4y\sqrt{y}}$$

$$D = A \cdot C - B^2$$

Then is :

$$D = (-2)\left(-\frac{x}{4y\sqrt{y}}\right) - \left(\frac{1}{2\sqrt{y}}\right)^2$$

$$\boxed{D = \frac{x}{2y\sqrt{y}} - \frac{1}{4y}}$$

The value of point  $M(4,4)$  replace in D :

$$D = \frac{x}{2y\sqrt{y}} - \frac{1}{4y}$$

$$D(4,4) = \frac{4}{2 \cdot 4\sqrt{4}} - \frac{1}{4 \cdot 4} = \frac{1}{4} - \frac{1}{16} = \frac{3}{16}$$

$$D(4,4) = \frac{3}{16} > 0$$

We conclude that our point  $M(4,4)$  is extreme.

As is  $A = -2$  , we conclude  $A = -2 < 0$  , and point  $M(4,4)$  is maximum!

Return to  $z = x\sqrt{y} - x^2 - y + 6x + 3$  to find the maximum value:

$$z = x\sqrt{y} - x^2 - y + 6x + 3$$

$$z(4,4) = 4\sqrt{4} - 4^2 - 4 + 6 \cdot 4 + 3$$

$$\boxed{z(4,4) = 15}$$

**Example 3.**

**Find the extreme values of function :**  $z = 3 \ln \frac{x}{6} + 2 \ln y + \ln(12 - x - y)$

**Solution:**

$$z = 3 \ln \frac{x}{6} + 2 \ln y + \ln(12 - x - y)$$

$$\frac{\partial z}{\partial x} = 3 \cdot \frac{1}{x} \cdot \frac{1}{6} + \frac{1}{12 - x - y} (-1) \rightarrow \boxed{\frac{\partial z}{\partial x} = \frac{3}{x} - \frac{1}{12 - x - y}}$$

$$\frac{\partial z}{\partial y} = 2 \cdot \frac{1}{y} + \frac{1}{12 - x - y} (-1) \rightarrow \boxed{\frac{\partial z}{\partial y} = \frac{2}{y} - \frac{1}{12 - x - y}}$$

Equate the first partial derivatives to zero and solve the system of equations:

$$\frac{3}{x} - \frac{1}{12-x-y} = 0$$

$$\frac{2}{y} - \frac{1}{12-x-y} = 0$$

$$\frac{3}{x} = \frac{1}{12-x-y} \rightarrow 4x + 3y = 36$$

$$\frac{2}{y} = \frac{1}{12-x-y} \rightarrow 2x + 3y = 24$$

$$4x + 3y = 36$$

$$2x + 3y = 24$$

$$x = 6$$

$$y = 4$$

$$\boxed{M(6,4)}$$

We got a stationary point.

We are looking for  $A = \frac{\partial^2 z}{\partial x^2}$ ,  $B = \frac{\partial^2 z}{\partial x \partial y}$ ,  $C = \frac{\partial^2 z}{\partial y^2}$  and  $D = A \cdot C - B^2$

$$\frac{\partial z}{\partial x} = \frac{3}{x} - \frac{1}{12-x-y}$$

$$\frac{\partial z}{\partial y} = \frac{2}{y} - \frac{1}{12-x-y}$$

$$A = \frac{\partial^2 z}{\partial x^2} = -\frac{3}{x^2} - \frac{1}{(12-x-y)^2}$$

$$B = \frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{(12-x-y)^2}$$

$$C = \frac{\partial^2 z}{\partial y^2} = -\frac{2}{y^2} - \frac{1}{(12-x-y)^2}$$

$$D = A \cdot C - B^2$$

$$D = \left(-\frac{3}{x^2} - \frac{1}{(12-x-y)^2}\right) \cdot \left(-\frac{2}{y^2} - \frac{1}{(12-x-y)^2}\right) - \frac{1}{(12-x-y)^4}$$

Substituting the value of fixed points in D:

$$M(6,4) \rightarrow D = \left(-\frac{3}{x^2} - \frac{1}{(12-x-y)^2}\right) \cdot \left(-\frac{2}{y^2} - \frac{1}{(12-x-y)^2}\right) - \frac{1}{(12-x-y)^4}$$

$$D(6,4) = \left(-\frac{3}{6^2} - \frac{1}{(12-6-4)^2}\right) \cdot \left(-\frac{2}{4^2} - \frac{1}{(12-6-4)^2}\right) - \frac{1}{(12-6-4)^4}$$

$$D(6,4) = \frac{1}{8} > 0$$

Now change this point in  $A$  to determine whether the max or min:

$$M(6,4) \rightarrow A = -\frac{3}{x^2} - \frac{1}{(12-x-y)^2}$$

$$A(6,4) = -\frac{3}{6^2} - \frac{1}{(12-6-4)^2} = -\frac{1}{12} - \frac{1}{4} < 0$$

Our point  $M(6,4)$  is therefore maximum!

The value function in it is:

$$z = 3 \ln \frac{x}{6} + 2 \ln y + \ln(12-x-y)$$

$$z(6,4) = 3 \ln \frac{6}{6} + 2 \ln 4 + \ln(12-6-4)$$

$$z(6,4) = 3 \ln 1 + 2 \ln 2^2 + \ln 2 = 3 \cdot 0 + 2 \cdot 2 \ln 2 + \ln 2$$

$$\boxed{z(6,4) = 5 \ln 2}$$