1. Examine function and draw a graph : $y = x^3 - 3x + 2$

Domain

$x \in (-\infty, \infty)$ or we can write $x \in R$

Zero function

$y = 0 \rightarrow x^3 - 3x + 2 = 0$

This is the equation of the third degree. In these situations we can use Bézout theorem or another "trick".

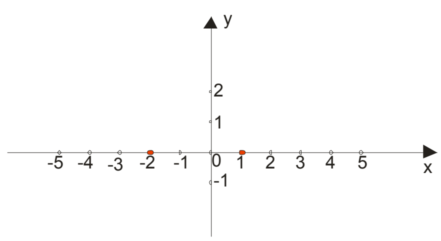
$$
\begin{aligned}
&x^3 - 3x + 2 = 0 \\
&x^3 - x - 2x + 2 = 0 \\
&x(x^2 - 1) - 2x + 2 = 0 \\
&x\underline{(x-1)}(x+1) - 2\underline{(x-1)} = 0 \\
&(x-1)[x(x+1) - 2] = 0 \\
&(x-1)[x^2 + x - 2] = 0
\end{aligned}
$$

As is for $x^2 + x - 2 = 0$ $x_1 = 1, x_2 = -2$ we will use $ax^2 + bx + c = a(x - x_1)(x - x_2)$ and

$$
\begin{aligned}
&(x-1)(x-1)(x+2) = 0 \\
&(x-1)^2(x+2) = 0
\end{aligned}
$$

Zero functions are therefore x = 1 and x = - 2

Points where graph cuts x axis:



Sign function

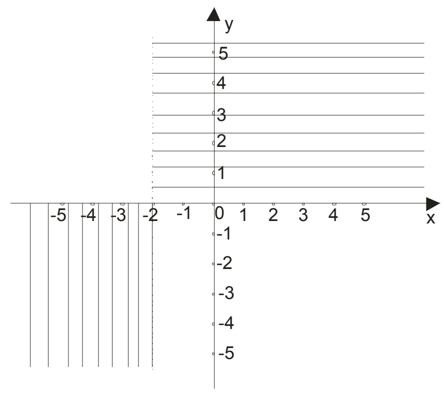
Consider the "signed" form function $y = (x-1)^2(x+2)$

From this we can conclude that $(x-1)^2 \geq 0$ and does not affect the sign function. So, sign depends only on the expression x+2 :

$y > 0$ for $x + 2 > 0$ that is for $x > -2$

$y < 0$ for $x + 2 < 0$ that is for $x < -2$

For graphics, it would mean:



Graph is only in these marked areas.

Parity

$f(-x) = (-x)^3 - 3(-x) + 2 = -x^3 + 3x + 2$

and this is $\neq$ f(x) and $\neq$ -f(x)

Extreme values (max and min) and monotonic function (increasing and decreasing)

$y = x^3 - 3x + 2$

$y` = 3x^2 - 3$

$y` = 0$

$3x^2 - 3 = 0$

$3(x-1)(x+1) = 0 \rightarrow x = 1 \vee x = -1$

<u>For x = -1</u>

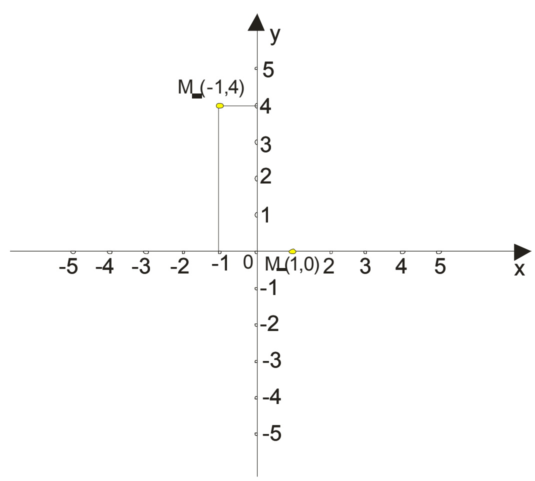
$y = (-1)^3 - 3 \cdot (-1) + 2 = -1 + 3 + 2 = 4$

We have point $M_1(-1, 4)$

<u>For x = 1</u>

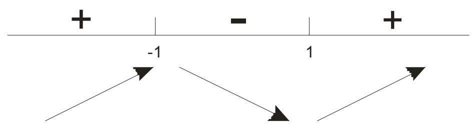
$y = 1^3 - 3 \cdot 1 + 2 = 0$

We have point $M_2(1, 0)$



we know that if y`> 0 the function is increasing, and if y `<0 function decreasing.

So is



function is **increasing** for $x \in (-\infty, -1) \cup (1, \infty)$

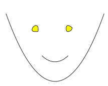
function **decreasing** for $x \in (-1, 1)$

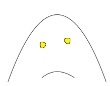
convexity and concavity

We are looking for second derivate :

$y` = 3x^2 - 3$
$y`` = 6x$
$y`` = 0$
$6x = 0 \rightarrow x = 0$
This change in the initial function value
<u>for x = 0</u>
$y = 0^3 - 3 \cdot 0 + 2$
$y = 2$
We have $P(0,2)$

We know that if y`` > 0 that is convex function (joyful ) and if y`` < 0 concave (sad)

y``>0 for 6x > 0 , x > 0 

y``<0 for 6x < 0 , x < 0 

Asymptote function (behavior functions at the ends of the field definition)

<u>Vertical asymptote</u>

There is not , because functions is defined everywhere ...

<u>Horizontal asymptote</u>

$\lim_{x\to\infty}(x^3 - 3x + 2) = \lim_{x\to\infty}(x-1)^2(x+2) = \infty \cdot \infty = \infty$
$\lim_{x\to-\infty}(x^3 - 3x + 2) = \lim_{x\to-\infty}(x-1)^2(x+2) = \infty \cdot (-\infty) = -\infty$
Therefore, we have not a horizontal asymptote

Oblique asymptote

$y = kx + n$

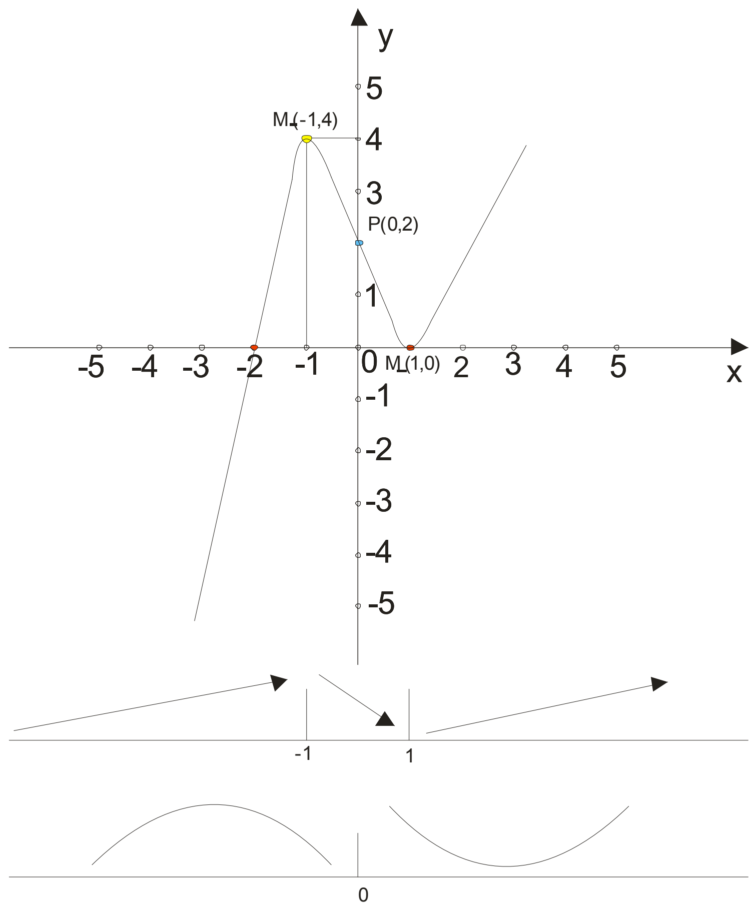
$$k = \lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty} \frac{x^3 - 3x + 2}{x} = \infty$$

do not have this asymptote

Sketch graphic

As we have seen any point in examining function tells us something about how the function looks like.

To draw the whole function now:



We suggest you to start the graphic below apply two straight lines parallel to where you first enter the results for

monotonic function and convexity and concavity.

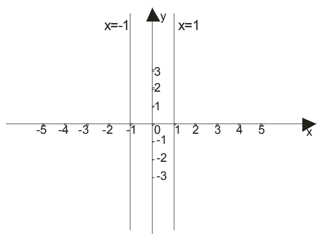
2. Examine function and draw a graph $y=\dfrac{x^2-4}{1-x^2}$

Domain

The function is defined for $1-x^2\neq 0$ then is $(1-x)(1+x)\neq 0 \;\rightarrow\; x\neq 1$ and $x\neq -1$

So $x\in(-\infty,-1)\cup(-1,1)\cup(1,\infty)$

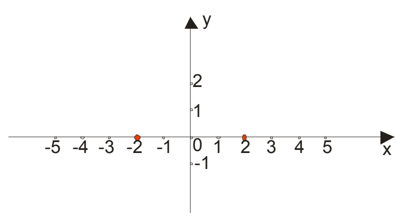
This tells us that the function is interrupted in x = -1 and x = 1



Zero function

y = 0 for $x^2-4=0\rightarrow(x-2)(x+2)=0\rightarrow x=2\vee x=-2$

Thus, the graph cuts x axis at two points -2 and 2



Sign function

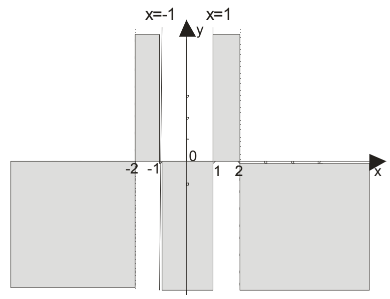
$y=\dfrac{x^2-4}{1-x^2}=\dfrac{(x-2)(x+2)}{(1-x)(1+x)}$ It is best to use the table …

	-∞ … -2	-2 … -1	-1 … 1	1 … 2	2 … ∞
x-2	−	−	−	−	+
x+2	−	+	+	+	+
1-x	+	+	+	−	−
1+x	−	−	+	+	+
y	−	+	−	+	−

Table tells us what?

It tells us where the graph **above the** x axis (where are +**)** and where it is **below the** x axis (where are **-)**

The picture would look like this:



Function exists only in shaded areas.

Parity

$$f(-x)=\frac{(-x)^2-4}{1-(-x)^2}=\frac{x^2-4}{1-x^2}=f(x)$$

Thus, the function is even, so the graph will be **symmetric with respect to the y-axis**

Extreme values (max and min) and monotonic function

$$y=\frac{x^2-4}{1-x^2}$$
$$y`=\frac{(x^2-4)`(1-x^2)-(1-x^2)`(x^2-4)}{(1-x^2)^2}$$
$$y`=\frac{2x(1-x^2)-(-2x)(x^2-4)}{(1-x^2)^2}$$
$$y`=\frac{2x(1-x^2)+2x(x^2-4)}{(1-x^2)^2}$$
$$y`=\frac{2x(1-x^2+x^2-4)}{(1-x^2)^2}$$
$$y`=\frac{-6x}{(1-x^2)^2}$$

y` = 0 for -6x = 0 , so x = 0 is point of extremes. When we replace x = 0 in the initial function, we have:

$$y=\frac{0^2-4}{1-0^2}=-4$$

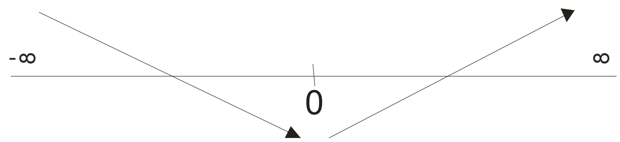
We got the point of extreme values **M(0,-4)**

We need sign of first derivative to examine monotonic function (increasing and decreasing).Let's think a bit ...

Expression in the denominator is always positive (because of the square), so that the sign of the first derivate affects only the term in the numerator.

So:

$$y` > 0 \to -6x > 0 \to x < 0$$
$$y` < 0 \to -6x < 0 \to x > 0$$



Point M(0,-4) is maximum.

convexity and concavity

$$y` = \frac{-6x}{(1-x^2)^2}$$

$$y`` = \frac{(-6x)`(1-x^2)^2 - ((1-x^2)^2)`(-6x)}{(1-x^2)^4}$$

$$y`` = \frac{-6(1-x^2)^2 - 2(1-x^2)(-2x)(-6x)}{(1-x^2)^4}$$

$$y`` = \frac{-6(1-x^2)^2 - 24x^2(1-x^2)}{(1-x^2)^4}$$

$$y`` = \frac{(1-x^2)[-6(1-x^2) - 24x^2]}{(1-x^2)^4}$$

$$y`` = \frac{-6 + 6x^2 - 24x^2}{(1-x^2)^3}$$

$$y`` = \frac{-6 - 18x^2}{(1-x^2)^3}$$

$$y`` = \frac{-6(1+3x^2)}{(1-x^2)^3}$$

y`` = 0 for $-6(3x^2+1) = 0$, And this has no rational solutions …

Convex and concave character is tested in the second derivative. Let's think back a bit ..

$3x^2 + 1 > 0$ and it does not affect the sign of the second derivative.

	$-\infty$ -1	1	∞
-6	−	−	−
1-x	+	+	−
1+x	−	+	+
y``	+	−	+
	∪	∩	∪

Asymptote function (behavior functions at the ends of the field definition)

<u>Vertical asymptote</u>

$$\lim_{\substack{x\to 1+\varepsilon,\\ \varepsilon\to 0}}\frac{x^2-4}{1-x^2}=\lim_{\substack{x\to 1+\varepsilon,\\ \varepsilon\to 0}}\frac{x^2-4}{(1-x)(1+x)}=\frac{1^2-4}{(1-(1+\varepsilon))(1+1+\varepsilon)}=\frac{-3}{(1-1-\varepsilon)2}=\frac{-3}{(-\varepsilon)2}=+\infty$$

$$\lim_{\substack{x\to 1-\varepsilon,\\ \varepsilon\to 0}}\frac{x^2-4}{1-x^2}=\lim_{\substack{x\to 1-\varepsilon,\\ \varepsilon\to 0}}\frac{x^2-4}{(1-x)(1+x)}=\frac{1^2-4}{(1-(1-\varepsilon))(1+1-\varepsilon)}=\frac{-3}{(1-1+\varepsilon)2}=\frac{-3}{\varepsilon 2}=-\infty$$

$$\lim_{\substack{x\to -1+\varepsilon,\\ \varepsilon\to 0}}\frac{x^2-4}{1-x^2}=\lim_{\substack{x\to -1+\varepsilon,\\ \varepsilon\to 0}}\frac{x^2-4}{(1-x)(1+x)}=\frac{(-1)^2-4}{(1-(-1+\varepsilon))(1+(-1+\varepsilon))}=\frac{-3}{(2-\varepsilon)\varepsilon}=\frac{-3}{2\varepsilon}=-\infty$$

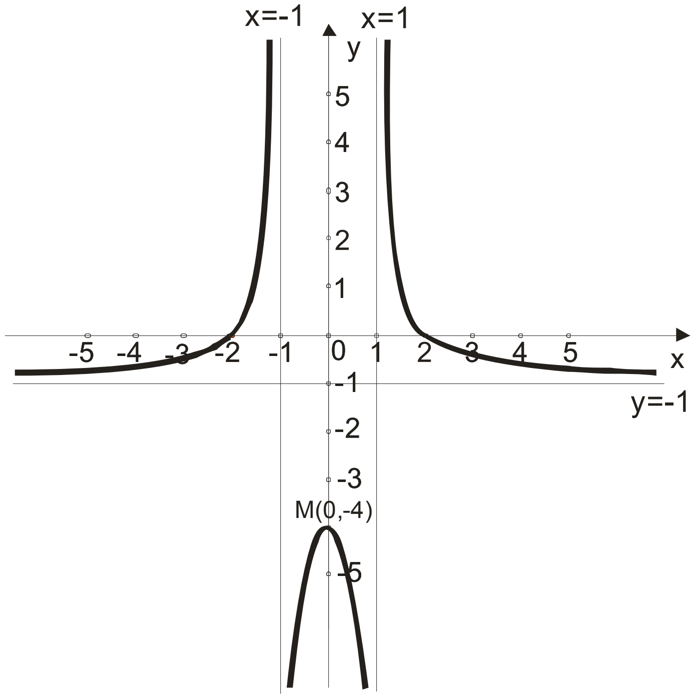
$$\lim_{\substack{x\to -1-\varepsilon,\\ \varepsilon\to 0}}\frac{x^2-4}{1-x^2}=\lim_{\substack{x\to -1-\varepsilon,\\ \varepsilon\to 0}}\frac{x^2-4}{(1-x)(1+x)}=\frac{(-1)^2-4}{(1-(-1-\varepsilon))(1+(-1-\varepsilon))}=\frac{-3}{(2+\varepsilon)(-\varepsilon)}=\frac{-3}{2(-\varepsilon)}=+\infty$$

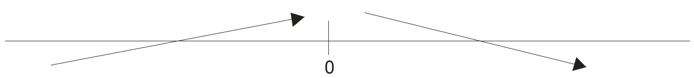
<u>Horizontal asymptote</u>

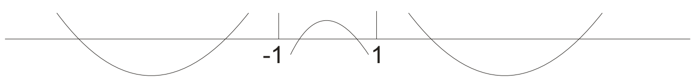
$\lim_{x\to\pm\infty}\frac{x^2-4}{1-x^2}=-\frac{1}{1}=-1$ so y <u>= - 1 is Horizontal asymptote</u>

This means that, because we have horizontal asymptote, we don`t have oblique asymptote.

the final graph:





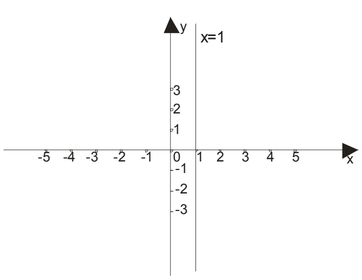


3. Examine function and draw a graph $y = \dfrac{x^2 - 4}{x - 1}$

Domain

The function is defined for $x - 1 \neq 0$ then is $x \neq 1$

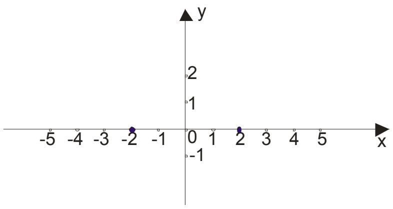
So: $x \in (-\infty, 1) \cup (1, \infty)$



Zero function

y = 0 for $x^2 - 4 = 0 \to (x-2)(x+2) = 0 \to x = 2 \vee x = -2$

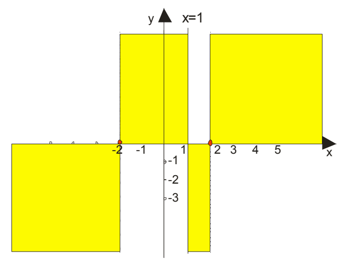
Thus, the graph cuts x axis at two points x = -2 and x = 2



Sign function

$$y = \frac{x^2 - 4}{x - 1} = \frac{(x-2)(x+2)}{x-1}$$

	-∞ -2	1	2	∞
x-2	−	−	−	+
x+2	−	+	+	+
x-1	−	−	+	+
y	−	+	−	+



This function is in yellow shaded areas.

Parity

$$f(-x) = \frac{(-x)^2 - 4}{-x - 1} = \frac{x^2 - 4}{-x - 1}$$

Extreme values (max and min) and monotonic function

$$y = \frac{x^2 - 4}{x - 1}$$

$$y` = \frac{(x^2-4)`(x-1) - (x-1)`(x^2-4)}{(x-1)^2}$$

$$y` = \frac{2x(x-1) - 1(x^2-4)}{(x-1)^2}$$

$$y` = \frac{2x^2 - 2x - 1x^2 + 4}{(x-1)^2} = \frac{x^2 - 2x + 4}{(x-1)^2}$$

$y` = 0$ for $x^2 - 2x + 4 = 0$

As is $x^2 - 2x + 4 > 0$ because $a > 0 \wedge D < 0$

We conclude that the function **has no extreme,** and is **constantly increasing. (** $y` > 0$ **)**

convexity and concavity

$$y` = \frac{x^2 - 2x + 4}{(x-1)^2}$$

$$y`` = \frac{(x^2 - 2x + 4)`(x-1)^2 - ((x-1)^2)`(x^2 - 2x + 4)}{(x-1)^4}$$

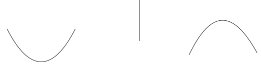
$$y`` = \frac{(2x-2)(x-1)^2 - 2(x-1)(x^2 - 2x + 4)}{(x-1)^4}$$

$$y`` = \frac{(x-1)[(2x-2)(x-1) - 2(x^2 - 2x + 4)]}{(x-1)^4}$$

$$y`` = \frac{[2x^2 - 2x - 2x + 2 - 2x^2 + 4x - 8]}{(x-1)^3}$$

$$y`` = \frac{-6}{(x-1)^3}$$

	-∞ 1	∞
-6	−	−
x-1	−	+
y``	+	−



Asymptote function (behavior functions at the ends of the field definition)

<u>Vertical asymptote</u>

$$\lim_{\substack{x \to 1+\varepsilon, \\ \varepsilon \to 0}} \frac{x^2 - 4}{x - 1} = \frac{1^2 - 4}{1 + \varepsilon - 1} = \frac{-3}{+\varepsilon} = \frac{-3}{+0} = -\infty$$

$$\lim_{\substack{x \to 1-\varepsilon, \\ \varepsilon \to 0}} \frac{x^2 - 4}{x - 1} = \frac{1^2 - 4}{1 - \varepsilon - 1} = \frac{-3}{-\varepsilon} = \frac{-3}{-0} = +\infty$$

Horizontal asymptote

$\lim_{x\to\pm\infty} \frac{x^2-4}{x-1} = \pm\infty$ This tells us that there is no horizontal asymptote and we ask for oblique asymptote.

Oblique asymptote

oblique asymptote **is line** $\boldsymbol{y = kx + n}$

$$k = \lim_{x\to\pm\infty} \frac{f(x)}{x} \qquad \text{and} \qquad n = \lim_{x\to\pm\infty} [f(x) - kx]$$

$$k = \lim_{x\to\pm\infty} \frac{\frac{x^2-4}{x-1}}{x} = \lim_{x\to\pm\infty} \frac{x^2-4}{x^2-x} = 1$$

$$n = \lim_{x\to\pm\infty} [f(x) - kx] = \lim_{x\to\pm\infty} \left[\frac{x^2-4}{x-1} - 1x\right] = \lim_{x\to\pm\infty} \left[\frac{x^2-4-x(x-1)}{x-1}\right] = \lim_{x\to\pm\infty} \left[\frac{x^2-4-x^2+x}{x-1}\right] = \lim_{x\to\pm\infty} \left[\frac{x-4}{x-1}\right] = 1$$

Replace k and n in : **y = kx + n** and we have **y = x + 1**