

**1. Examine function and draw a graph :**  $y = \sqrt[3]{x^2 - x^3}$

**Domain**

This function is defined everywhere because there are no fractions and the third root is everywhere defined

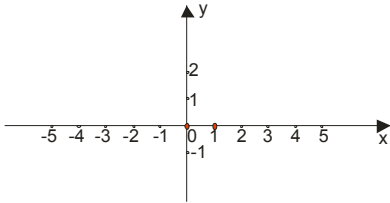
So:  $D_f = (-\infty, \infty)$

This tells us that there is no vertical asymptote.

**Zero function**

$$y = 0 \rightarrow x^2 - x^3 = 0 \rightarrow x^2(1 - x) = 0$$

$$x = 0; x = 1$$



**Sign function**

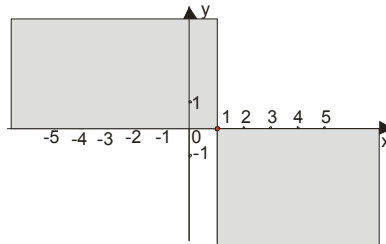
As always, think first of which depends on sign function ?

$$y = \sqrt[3]{x^2 - x^3} = \sqrt[3]{x^2(1 - x)}$$

Only  $1 - x$ , then is :

	$-\infty$	1	$\infty$
$1 - x$	+	-	
$y$	+	-	

the diagram is



**Parity**

$$f(-x) = \sqrt[3]{(-x)^2 - (-x)^3} = \sqrt[3]{x^2 + x^3}$$

**Extreme values (max and min) and monotonic function ( increasing and decreasing )**

$y = \sqrt[3]{x^2 - x^3}$  easier for us to ask if function look like :

$$y = (x^2 - x^3)^{\frac{1}{3}}$$

$$y' = \frac{1}{3}(x^2 - x^3)^{\frac{1}{3}-1} \cdot (x^2 - x^3)'$$

$$y' = \frac{1}{3}(x^2 - x^3)^{-\frac{2}{3}} \cdot (2x - 3x^2)$$

$$y' = \frac{1}{3} \frac{2x - 3x^2}{\sqrt[3]{(x^2 - x^3)^2}}$$

$$y' = 0 \rightarrow 2x - 3x^2 = 0 \rightarrow x(2 - 3x) = 0 \rightarrow x = 0 \vee x = \frac{2}{3}$$

For  $x = 0$

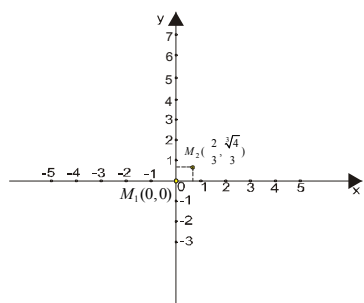
$$y = \sqrt[3]{0 - 0} = 0$$

For  $x = \frac{2}{3}$

$$y = \sqrt[3]{\left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^3} = \sqrt[3]{\frac{4}{9} - \frac{8}{27}} = \sqrt[3]{\frac{4}{27}} = \frac{\sqrt[3]{4}}{3}$$

So:

$$M_1(0, 0); M_2\left(\frac{2}{3}, \frac{\sqrt[3]{4}}{3}\right)$$



	$-\infty$	0	$\frac{2}{3}$	$\infty$
x	-	+	+	
2-3x	+	+	-	
y'	-	+	-	

↘ ↗ ↘

## convexity and concavity

$$y = \sqrt[3]{x^2 - x^3}$$

$$y' = \frac{1}{3}(x^2 - x^3)^{-\frac{2}{3}} \cdot (2x - 3x^2)$$

$$y'' = \frac{1}{3} \left[ \left( (x^2 - x^3)^{-\frac{2}{3}} \right)' \cdot (2x - 3x^2) + (x^2 - x^3)^{-\frac{2}{3}} \cdot (2x - 3x^2)' \right]$$

$$y'' = \frac{1}{3} \left[ -\frac{2}{3}(x^2 - x^3)^{-\frac{5}{3}} \cdot (2x - 3x^2) \cdot (2x - 3x^2) + (x^2 - x^3)^{-\frac{2}{3}} \cdot (2 - 6x) \right]$$

After careful calculation :

$$y'' = \frac{1}{3} \frac{-2}{\sqrt[3]{x^2(1-x)^4}}$$

## Asymptote function (behavior functions at the ends of the field definition)

As we have already concluded, the function has no vertical asymptote!

### Horizontal asymptote

$$\lim_{x \rightarrow +\infty} \sqrt[3]{x^2 - x^3} = \lim_{x \rightarrow +\infty} \sqrt[3]{x^3 \left( \frac{1}{x} - 1 \right)} = \lim_{x \rightarrow +\infty} x \cdot \lim_{x \rightarrow +\infty} \sqrt[3]{\left( \frac{1}{x} - 1 \right)} = \infty \cdot (-1) = -\infty$$

$$\lim_{x \rightarrow -\infty} \sqrt[3]{x^2 - x^3} = \lim_{x \rightarrow -\infty} \sqrt[3]{x^3 \left( \frac{1}{x} - 1 \right)} = \lim_{x \rightarrow -\infty} x \cdot \lim_{x \rightarrow -\infty} \sqrt[3]{\left( \frac{1}{x} - 1 \right)} = -\infty \cdot (-1) = +\infty$$

### Oblique asymptote

$$k = \lim_{x \rightarrow \infty} \frac{\sqrt[3]{x^2 - x^3}}{x} = \lim_{x \rightarrow \infty} \frac{\sqrt[3]{x^3 \left( \frac{1}{x} - 1 \right)}}{x} = \lim_{x \rightarrow \infty} \frac{\cancel{x} \cdot \sqrt[3]{\left( \frac{1}{x} - 1 \right)}}{\cancel{x}} = \lim_{x \rightarrow \infty} \sqrt[3]{\left( \frac{1}{x} - 1 \right)} = 0 - 1 = -1$$

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$$n = \lim_{x \rightarrow \infty} [f(x) - kx]$$

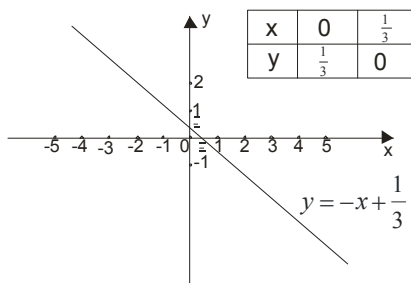
$$n = \lim_{x \rightarrow \infty} [\sqrt[3]{x^2 - x^3} + x] \quad \text{here we must use : } A^3 + B^3 = (A + B)(A^2 - AB + B^2)$$

$$n = \lim_{x \rightarrow \infty} (\sqrt[3]{x^2 - x^3} + x) \cdot \frac{(\sqrt[3]{x^2 - x^3})^2 - \sqrt[3]{x^2 - x^3} \cdot x + x^2}{(\sqrt[3]{x^2 - x^3})^2 - \sqrt[3]{x^2 - x^3} \cdot x + x^2} = \lim_{x \rightarrow \infty} \frac{(\sqrt[3]{x^2 - x^3})^3 + x^3}{(\sqrt[3]{x^2 - x^3})^2 - \sqrt[3]{x^2 - x^3} \cdot x + x^2}$$

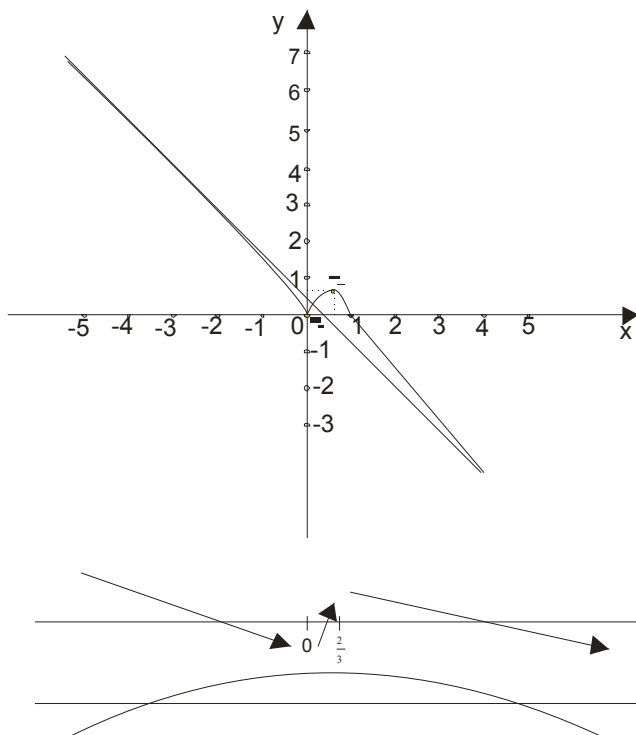
$$n = \lim_{x \rightarrow \infty} \frac{x^2 - x^3 + x^3}{\sqrt[3]{x^4 - 2x^5 + x^6} - \sqrt[3]{x^3(1 - \frac{1}{x})} \cdot x + x^2} = \lim_{x \rightarrow \infty} \frac{x^2}{\sqrt[3]{x^6(\frac{1}{x^2} - \frac{2}{x} + 1)} - x \cdot \sqrt[3]{(1 - \frac{1}{x})} \cdot x + x^2}$$

$$n = \lim_{x \rightarrow \infty} \frac{x^2}{x^2 [\sqrt[3]{(\frac{1}{x^2} - \frac{2}{x} + 1)} - \sqrt[3]{(1 - \frac{1}{x})} + 1]} = \frac{1}{1 - (-1) + 1} = \frac{1}{3}$$

$$y = -x + \frac{1}{3}$$



And finally:



**2. Examine function and draw a graph:**  $y = \frac{x-2}{\sqrt{x^2+2}}$

**Domain**

Here we observe two conditions:

$$\sqrt{x^2+2} \neq 0 \quad \text{and} \quad x^2+2 \geq 0$$

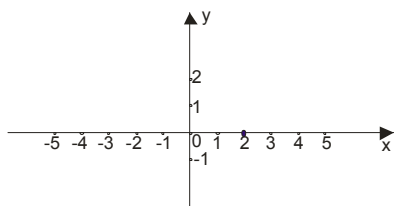
$$\text{So: } x^2+2 > 0$$

This is obviously true for any real  $x$ , so:  $D_f = (-\infty, \infty)$

And here we conclude that the function has no vertical asymptote.

**Zero function**

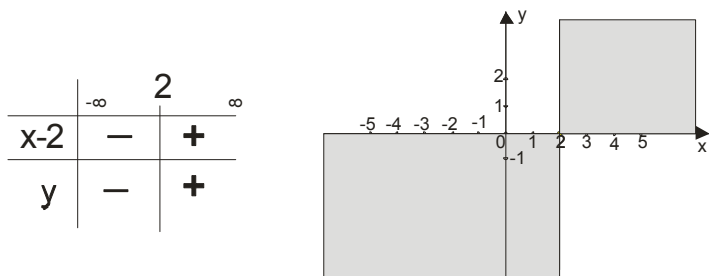
$$y=0 \quad \text{for } x-2 = 0, \text{ then we have: } x = 2$$



**Sign function**

$$y = \frac{x-2}{\sqrt{x^2+2}}$$

Kako je izraz u imeniocu uvek pozitivan, zaključujemo da znak zavisi samo od brojioca...



**Parity**

$$f(-x) = \frac{-x-2}{\sqrt{(-x)^2+2}} = \frac{-x-2}{\sqrt{x^2+2}} \neq f(x)$$

**Extreme values (max and min) and monotonic function ( increasing and decreasing )**

$$y = \frac{x-2}{\sqrt{x^2+2}}$$

$$y' = \frac{(x-2) \cdot \sqrt{x^2+2} - (\sqrt{x^2+2}) \cdot (x-2)}{(\sqrt{x^2+2})^2}$$

$$y' = \frac{1 \cdot \sqrt{x^2+2} - \frac{1}{2\sqrt{x^2+2}} \cdot (x^2+2) \cdot (x-2)}{x^2+2}$$

$$y' = \frac{1 \cdot \sqrt{x^2+2} - \frac{1}{2\sqrt{x^2+2}} \cdot 2x \cdot (x-2)}{x^2+2}$$

$$y' = \frac{(\sqrt{x^2+2})^2 - x(x-2)}{x^2+2}$$

$$y' = \frac{x^2+2 - x^2+2x}{(x^2+2)\sqrt{x^2+2}}$$

$$y' = \frac{2+2x}{(x^2+2)\sqrt{x^2+2}}$$

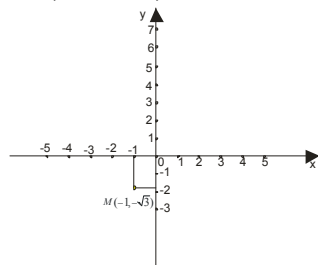
$$y' = \frac{2(x+1)}{(x^2+2)\sqrt{x^2+2}} \rightarrow \rightarrow \rightarrow y' = \frac{2(x+1)}{(x^2+2)^{\frac{3}{2}}}$$

$y'=0$  for  $x+1=0$ , and then  $x=-1$

For  $x=-1$

$$y = \frac{-1-2}{\sqrt{1+2}} = \frac{-3}{\sqrt{3}} = \frac{-3}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = -\sqrt{3}$$

$M(-1, -\sqrt{3})$



	$-\infty$	$-1$	$\infty$
$2x+2$	$-$	$+$	
$y'$	$-$	$+$	
	↘		↗

M is minimum point.

## convexity and concavity

$$y = \frac{x-2}{\sqrt{x^2+2}}$$

$$y' = \frac{2(x+1)}{(x^2+2)^{\frac{3}{2}}}$$

$$y'' = 2 \frac{(x+1) \cdot (x^2+2)^{\frac{3}{2}} - ((x^2+2)^{\frac{3}{2}})' \cdot (x+1)}{((x^2+2)^{\frac{3}{2}})^2}$$

$$y'' = 2 \frac{1 \cdot (x^2+2)^{\frac{3}{2}} - \frac{3}{2}(x^2+2)^{\frac{3}{2}-1} (x^2+2)' \cdot (x+1)}{(x^2+2)^3}$$

$$y'' = 2 \frac{(x^2+2)^{\frac{3}{2}} - \cancel{3} \cdot \cancel{2} x \cdot (x+1)}{(x^2+2)^3}$$

$$y'' = 2 \frac{(x^2+2)^{\frac{3}{2}} - 3(x^2+2)^{\frac{1}{2}} \cdot x \cdot (x+1)}{(x^2+2)^3}$$

$$y'' = 2 \frac{\cancel{(x^2+2)^{\frac{1}{2}}} [x^2+2-3x \cdot (x+1)]}{(x^2+2)^3}$$

$$y'' = 2 \frac{x^2+2-3x^2-3x}{(x^2+2)^{\frac{5}{2}}}$$

$$y'' = 2 \frac{-2x^2-3x+2}{(x^2+2)^{\frac{5}{2}}}$$

$$y'' = 0$$

$$-2x^2-3x+2=0 \rightarrow x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \rightarrow x_1 = -2 \wedge x_2 = \frac{1}{2}$$

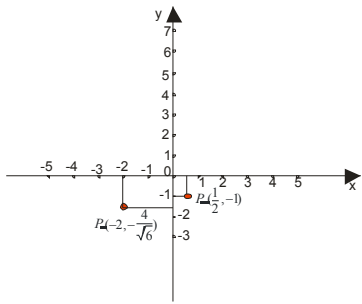
$$\text{For } x_1 = -2 \rightarrow y_1 = \frac{-4}{\sqrt{6}}$$

$$\text{For } x_2 = \frac{1}{2} \rightarrow y_2 = -1$$

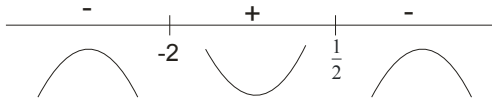
We have two points:

$$P_1(-2, \frac{-4}{\sqrt{6}})$$

$$P_2(\frac{1}{2}, -1)$$



Sign of second derivate, again, depends only on the terms in the numerator  $-2x^2 - 3x + 2$ .



### ***Asymptote function (behavior functions at the ends of the field definition)***

As we have already concluded, the function has no vertical asymptote!

### Horizontal asymptote

$$\lim_{x \rightarrow \pm\infty} \frac{x-2}{\sqrt{x^2+2}} = \lim_{x \rightarrow \pm\infty} \frac{x-2}{\sqrt{x^2(1+\frac{2}{x^2})}} = \lim_{x \rightarrow \pm\infty} \frac{x-2}{|x|\sqrt{1+\frac{2}{x^2}}}$$

Watch out! As we get down absolute value ,

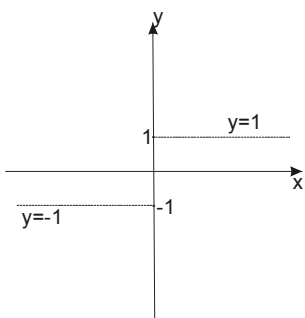
we must separate limit for the + and - infinity!

$$\lim_{x \rightarrow +\infty} \frac{x-2}{x\sqrt{1+\frac{2}{x^2}}} = 1$$

$$\lim_{x \rightarrow -\infty} \frac{x-2}{-x\sqrt{1+\frac{2}{x^2}}} = -1$$

If x approaches + infinite horizontal asymptote is  $y = 1$

If x approaches - infinity horizontal asymptote is  $y = -1$



The final graph is:

