

## 1. Examine function and draw a graph : $y = xe^x$

### *Domain*

This function is everywhere defined , because  $e^x$  is defined for *all*  $x$  in the set  $\mathbb{R}$ .

So:  $x \in (-\infty, \infty)$  . This immediately tells us that the function **has no vertical asymptote!**

### *Zero function*

$$y = 0$$

$$xe^x = 0 \rightarrow x = 0$$

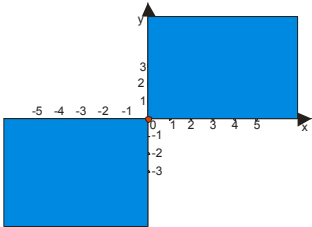
To remind you that  $e^x > 0$  always.

### *Sign function*

$$y > 0 \rightarrow xe^x > 0 \rightarrow x > 0$$

$$y < 0 \rightarrow xe^x < 0 \rightarrow x < 0$$

The diagram would look:



The function is only in blue field and the x-axis cuts only in  $x = 0$ .

### *Parity*

$$f(-x) = -xe^{-x} = \frac{-x}{e^x} \neq f(x)$$

This tells us that the function is neither even nor odd.

**Extreme values (max and min) and monotonic function ( increasing and decreasing )**

$$y = xe^x$$

$$y' = x'e^x + (e^x)'x$$

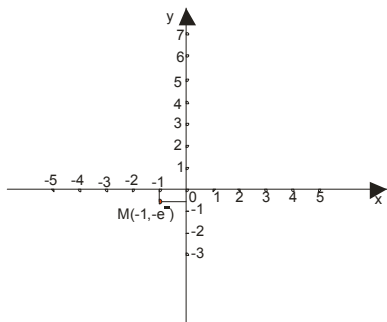
$$y' = 1e^x + e^x x$$

$$y' = e^x(1+x)$$

$$y' = 0 \rightarrow e^x(1+x) \rightarrow 1+x = 0 \rightarrow x = -1$$

$$\text{For } x = -1 \text{ is } y = (-1)e^{-1} \rightarrow y = -\frac{1}{e}$$

Thus, the extreme point is  $M(-1, -\frac{1}{e})$



What the sign of the first derivate depends on ?

As is  $e^x > 0$  always , the sign of the first derivate depends on  $1+x$

	$-\infty$	$-1$	$\infty$
$1+x$	$-$		$+$
$y'$	$-$		$+$

$\swarrow$                        $\searrow$

Point M is then the minimum point.

**convexity and concavity**

$$y' = e^x(1+x)$$

$$y'' = (e^x)'(1+x) + (1+x)'e^x$$

$$y'' = e^x(1+x) + e^x$$

$$y'' = e^x(x+2)$$

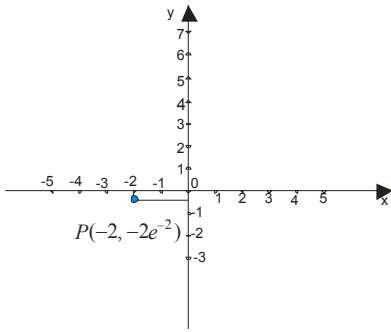
$$y'' = 0$$

$$x+2 = 0 \rightarrow x = -2$$

$$\text{For } x = -2 \text{ is } y = -2e^{-2} = \frac{-2}{e^2}$$



We have point  $P(-2, -2e^{-2})$ .

Find that approximately  $-2e^{-2} \approx -0,27$



What the sign of the second derivative depends on ?

As  $e^x > 0$  then depends on  $x + 2$

	$-\infty$	$-2$	$\infty$
$x+2$	-	+	
$y''$	- 	+	

**Asymptote function (behavior functions at the ends of the field definition)**

As we have said, there is no Vertical asymptote.

Horizontal asymptote

One small tip: If you have a function  $e^x$ , especially the work of  $x \rightarrow +\infty$  and then  $x \rightarrow -\infty$ , because:

$$e^{\infty} = \infty$$

$$e^{-\infty} = 0$$

So:

$$\lim_{x \rightarrow +\infty} xe^x = \infty \cdot e^{\infty} = \infty \cdot \infty = \infty$$

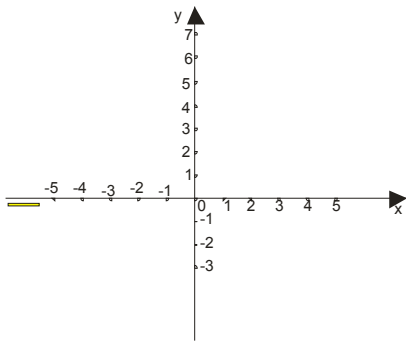
$$\lim_{x \rightarrow -\infty} xe^x = -\infty \cdot e^{-\infty} = -\infty \cdot 0 = ?$$

$$\lim_{x \rightarrow -\infty} xe^x = \lim_{x \rightarrow -\infty} \frac{x}{e^{-x}} = \frac{-\infty}{\infty} = \frac{-\infty}{\infty} = l' \text{ H\^o}pital = \lim_{x \rightarrow -\infty} \frac{1}{-e^{-x}} = \frac{1}{-\infty} = 0_-$$

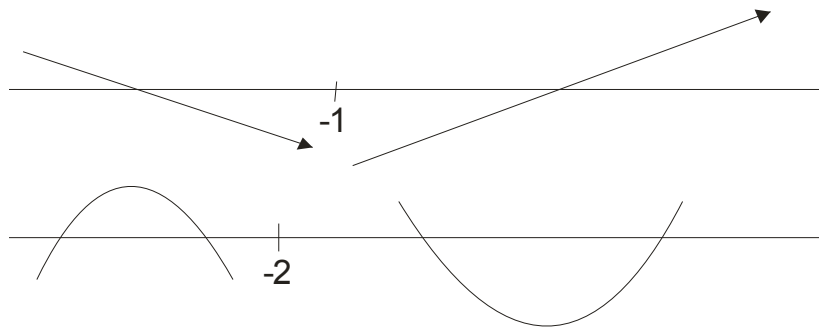
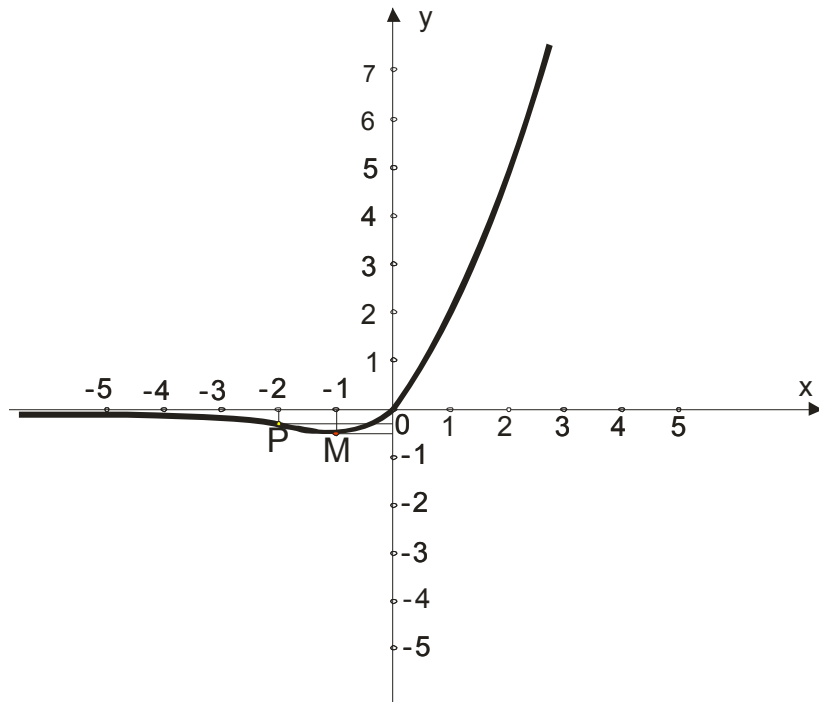
What does this tell us?

When  $x \rightarrow +\infty$ , there is no horizontal asymptote, but when  $x \rightarrow -\infty$  horizontal asymptote is  $y=0$ , ie,

When  $x$  approaches  $-\infty$ , the function approaches zero from below, from the negative side!



And to make the final graph:



## 2. Examine function and draw a graph : $y = \frac{e^x}{x}$

### *Domain*

$$x \neq 0 \rightarrow x \in (-\infty, 0) \cup (0, \infty)$$

This means that the function in  $x = 0$  has the potential vertical asymptote

### *Zero function*

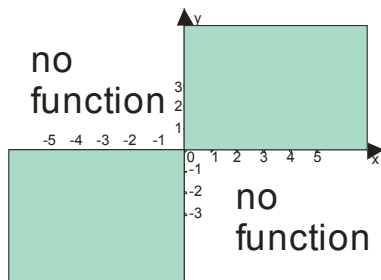
As we have said  $e^x > 0$ , so the function has no zero (nowhere cuts x axis)

### *Sign function*

It is clear that the sign functions depend only on x

$$y > 0 \rightarrow x > 0$$

$$y < 0 \rightarrow x < 0$$



### *Parity*

$$f(-x) = \frac{e^{-x}}{-x} = -\frac{1}{xe^x} \neq f(x)$$

**Extreme values (max and min) and monotonic function ( increasing and decreasing )**

$$y = \frac{e^x}{x}$$

$$y' = \frac{(e^x)'x - x'e^x}{x^2}$$

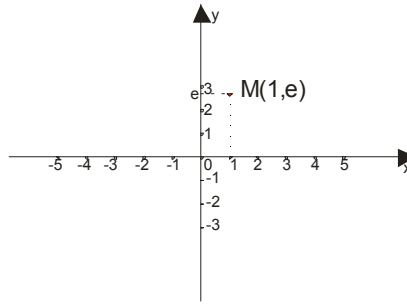
$$y' = \frac{e^x x - 1e^x}{x^2}$$

$$y' = \frac{e^x(x-1)}{x^2}$$

$$y' = 0 \rightarrow e^x(x-1) = 0 \rightarrow x-1 = 0 \rightarrow x = 1$$

For  $x=1$  is  $y = \frac{e^1}{1} \rightarrow y = e$

$M(1, e)$  is extreme point



As is  $x^2 > 0$  and  $e^x > 0$  we conclude that the sign of the first derivate depends only on  $x-1$

	$-\infty$	1	$\infty$
$x-1$	-	+	
$y'$	-	+	

↙ ↘

Point M is then the minimum point!

**convexity and concavity**

$$y' = \frac{e^x(x-1)}{x^2}$$

$$y'' = \frac{[e^x(x-1)]' \cdot x^2 - (x^2)' \cdot e^x(x-1)}{x^4}$$

$$y'' = \frac{[(e^x)'(x-1) + (x-1)'e^x] \cdot x^2 - 2x \cdot e^x(x-1)}{x^4}$$

$$y'' = \frac{[e^x(x-1) + 1e^x] \cdot x^2 - 2x \cdot e^x(x-1)}{x^4}$$

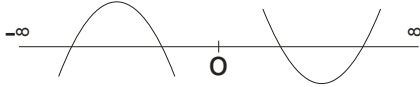
$$y'' = \frac{[e^x x - 1e^x + 1e^x] \cdot x^2 - 2x \cdot e^x(x-1)}{x^4} = \frac{e^x x \cdot x^2 - 2x \cdot e^x(x-1)}{x^4} = \frac{e^x \cancel{x} \cdot (x^2 - 2(x-1))}{x^4}$$

$$y'' = \frac{e^x(x^2 - 2x + 2)}{x^3}$$

$$y'' = 0 \rightarrow x^2 - 2x + 2 = 0$$

This quadratic equation has no solution, because it is  $D < 0$  and  $a > 0$ .

So:  $x^2 - 2x + 2 > 0$

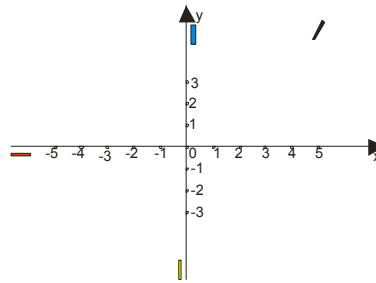


***Asymptote function (behavior functions at the ends of the field definition)***

Vertical asymptote

$$\lim_{x \rightarrow 0+\varepsilon} \frac{e^x}{x} = \frac{e^0}{0+\varepsilon} = \frac{1}{+\varepsilon} = +\infty \quad (\text{Blue line})$$

$$\lim_{x \rightarrow 0-\varepsilon} \frac{e^x}{x} = \frac{e^0}{0-\varepsilon} = \frac{1}{-\varepsilon} = -\infty \quad (\text{Yellow line})$$



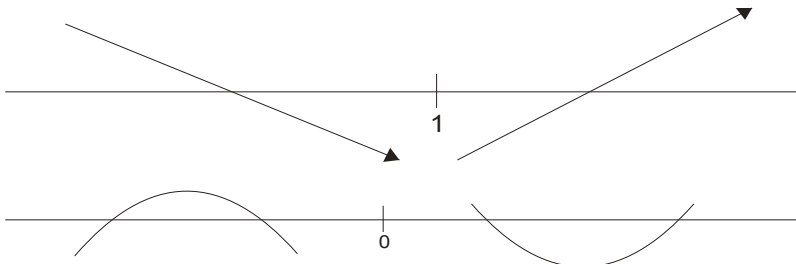
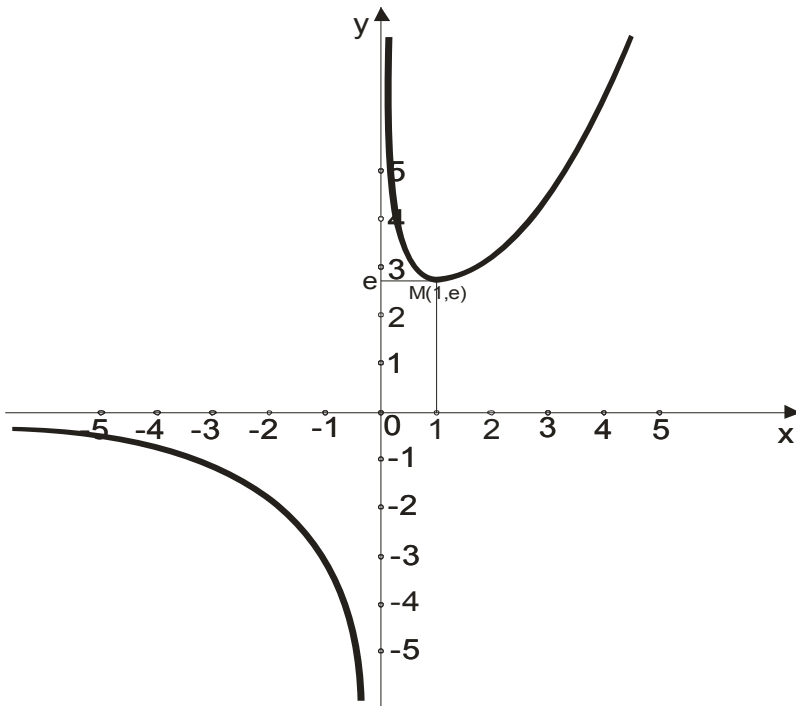
Horizontal asymptote

$$\lim_{x \rightarrow +\infty} \frac{e^x}{x} = \frac{e^\infty}{\infty} = \frac{\infty}{\infty} = l'H\acute{o}pital = \lim_{x \rightarrow +\infty} \frac{(e^x)'}{x'} = \lim_{x \rightarrow +\infty} \frac{e^x}{1} = e^\infty = \infty \quad (\text{Black line})$$

$$\lim_{x \rightarrow -\infty} \frac{e^x}{x} = \frac{e^{-\infty}}{-\infty} = \frac{0}{-\infty} = 0_- \quad (\text{Red line})$$

Therefore, the function has a horizontal asymptote  $y = 0$  but only on the left

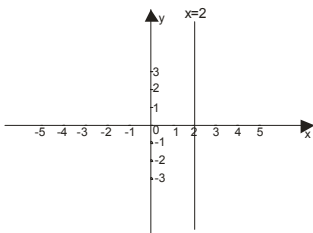
The final graph looks like:



3. Examine function and draw a graph :  $y = x \cdot e^{\frac{1}{x-2}}$

**Domain**

$$x - 2 \neq 0 \rightarrow x \neq 2 \rightarrow x \in (-\infty, 2) \cup (2, \infty)$$



## Zero function

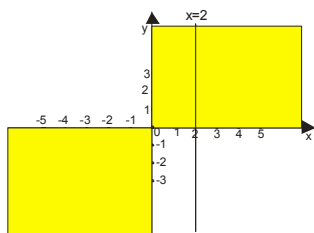
$$y = 0 \rightarrow x \cdot e^{\frac{1}{x-2}} = 0 \rightarrow x = 0 \text{ because } e^{\frac{1}{x-2}} > 0 \text{ always}$$

## Sign function

As is  $e^{\frac{1}{x-2}} > 0$ , concludes that the sign function depends only on  $x-2$

$y > 0$  when  $x - 2 > 0$ , then  $x > 2$

$y < 0$  when  $x - 2 < 0$ , then  $x < 2$



The function is only found in the yellow areas.

## Parity

$$f(-x) = -x \cdot e^{\frac{1}{-x-2}} \neq f(x)$$

## Extreme values (max and min) and monotonic function (increasing and decreasing)

$$y = x \cdot e^{\frac{1}{x-2}} \dots\dots\dots (e^{\Theta})' = e^{\Theta} \cdot \Theta'$$

$$y' = 1 \cdot e^{\frac{1}{x-2}} + (e^{\frac{1}{x-2}})' \cdot x$$

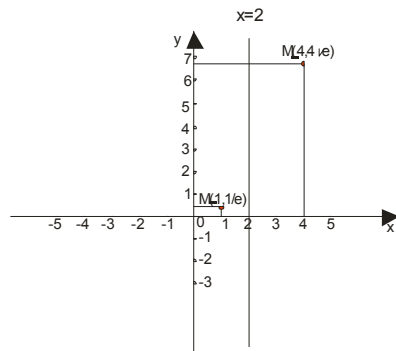
$$y' = e^{\frac{1}{x-2}} + e^{\frac{1}{x-2}} \cdot \left(\frac{1}{x-2}\right)' \cdot x$$

$$y' = e^{\frac{1}{x-2}} + e^{\frac{1}{x-2}} \cdot \left(-\frac{1}{(x-2)^2}\right) \cdot x = e^{\frac{1}{x-2}} - e^{\frac{1}{x-2}} \cdot \frac{1}{(x-2)^2} \cdot x = e^{\frac{1}{x-2}} \cdot \left(1 - \frac{x}{(x-2)^2}\right) = e^{\frac{1}{x-2}} \cdot \frac{(x-2)^2 - x}{(x-2)^2}$$

$$y' = e^{\frac{1}{x-2}} \cdot \frac{x^2 - 4x + 4 - x}{(x-2)^2}$$

$$y' = e^{\frac{1}{x-2}} \cdot \frac{x^2 - 5x + 4}{(x-2)^2}$$

$$y' = 0 \rightarrow x^2 - 5x + 4 = 0 \rightarrow x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \rightarrow x_1 = 1; x_2 = 4$$



For  $x = 1$

$$y = 1 \cdot e^{\frac{1}{1-2}} = e^{-1} = \frac{1}{e} \rightarrow M_1 = \left(1, \frac{1}{e}\right)$$

For  $x = 4$

$$y = 4 \cdot e^{\frac{1}{4-2}} = 4e^{\frac{1}{2}} = 4\sqrt{e} \rightarrow M_2 = (4, 4\sqrt{e})$$

What the sign of the first derivate depends on ?

As is  $e^{\frac{1}{x-2}} > 0$  **and**  $(x-2)^2 > 0$ , sign of the first derivate depends on  $x^2 - 5x + 4$

	$-\infty$	1	4	$\infty$
$x-1$	-	+	+	
$x-4$	-	-	+	
$y'$	+	-	+	

**convexity and concavity**

$$y' = e^{\frac{1}{x-2}} \cdot \frac{x^2 - 5x + 4}{(x-2)^2}$$

$$y'' = (e^{\frac{1}{x-2}})' \cdot \frac{x^2 - 5x + 4}{(x-2)^2} + (e^{\frac{1}{x-2}}) \cdot \left(\frac{x^2 - 5x + 4}{(x-2)^2}\right)'$$

$$y'' = e^{\frac{1}{x-2}} \cdot \left(-\frac{1}{(x-2)^2}\right) \cdot \frac{x^2 - 5x + 4}{(x-2)^2} + \frac{(x^2 - 5x + 4)' \cdot (x-2)^2 - ((x-2)^2)' \cdot (x^2 - 5x + 4)}{(x-2)^4} \cdot e^{\frac{1}{x-2}}$$

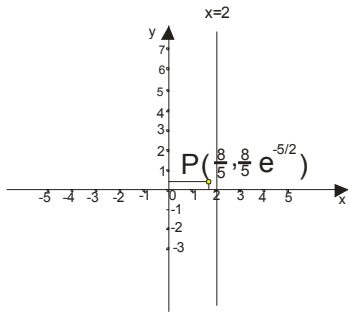
We get:

$$y'' = \frac{5x - 8}{(x-2)^4} \cdot e^{\frac{1}{x-2}}$$

$$y'' = 0 \rightarrow 5x - 8 = 0 \rightarrow x = \frac{8}{5}$$



$$\text{For } x = \frac{8}{5} \rightarrow y = \frac{8}{5} \cdot e^{\frac{1}{\frac{8}{5}-2}} \rightarrow y = \frac{8}{5} \cdot e^{-\frac{1}{2}} \rightarrow y = \frac{8}{5} \cdot e^{-\frac{5}{2}}$$

$$\text{So : } P\left(\frac{8}{5}, \frac{8}{5} \cdot e^{-\frac{5}{2}}\right)$$



What the sign of the second derivate depends on ?

Sign of the second derivate depends on  $5x-8$

	$-\infty$	$\frac{8}{5}$	$\infty$
$5x-8$	-		+
$y''$	-		+
			

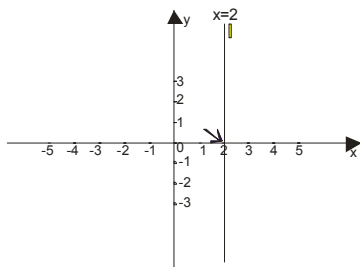
**Asymptote function (behavior functions at the ends of the field definition)**

Vertical asymptote

$$y = x \cdot e^{\frac{1}{x-2}}$$

$$\lim_{x \rightarrow 2+\varepsilon} x e^{\frac{1}{x-2}} = 2 \cdot e^{\frac{1}{2+\varepsilon-2}} = 2 \cdot e^{+\varepsilon} = 2 \cdot e^{\infty} = \infty \quad (\text{Yellow line})$$

$$\lim_{x \rightarrow 2-\varepsilon} x e^{\frac{1}{x-2}} = 2 \cdot e^{\frac{1}{2-\varepsilon-2}} = 2 \cdot e^{-\varepsilon} = 2 \cdot e^{-\infty} = 2 \cdot 0 = 0 \quad (\text{blue arrow})$$



## Horizontal asymptote

$$\lim_{x \rightarrow +\infty} x e^{\frac{1}{x-2}} = \infty \cdot e^{\frac{1}{\infty-2}} = \infty \cdot e^{\frac{1}{\infty}} = \infty \cdot e^0 = \infty \cdot 1 = \infty$$

$$\lim_{x \rightarrow -\infty} x e^{\frac{1}{x-2}} = -\infty \cdot e^{\frac{1}{-\infty-2}} = -\infty \cdot e^{\frac{1}{-\infty}} = -\infty \cdot e^0 = -\infty \cdot 1 = -\infty$$

No horizontal asymptotes, and we must examine whether there is oblique asymptote ...

## Oblique asymptote

$$y = kx + n$$

$$k = \lim_{x \rightarrow \pm\infty} \frac{f(x)}{x} = \lim_{x \rightarrow \pm\infty} \frac{\cancel{x} e^{\frac{1}{x-2}}}{\cancel{x}} = \lim_{x \rightarrow \pm\infty} e^{\frac{1}{x-2}} = e^{\frac{1}{\infty-2}} = e^0 = 1$$

$$n = \lim_{x \rightarrow \pm\infty} [f(x) - kx] = \lim_{x \rightarrow \pm\infty} [x e^{\frac{1}{x-2}} - 1 \cdot x] = \lim_{x \rightarrow \pm\infty} x(e^{\frac{1}{x-2}} - 1) = \infty \cdot 0 = ?$$

$$= \lim_{x \rightarrow \pm\infty} \frac{e^{\frac{1}{x-2}} - 1}{\frac{1}{x}} = \frac{0}{0} = \lim_{x \rightarrow \pm\infty} \frac{e^{\frac{1}{x-2}} \cdot \left(-\frac{1}{(x-2)^2}\right)}{-\frac{1}{x^2}} = \lim_{x \rightarrow \pm\infty} e^{\frac{1}{x-2}} \cdot \frac{x^2}{(x-2)^2} = \lim_{x \rightarrow \pm\infty} e^{\frac{1}{x-2}} \cdot \lim_{x \rightarrow \pm\infty} \frac{x^2}{(x-2)^2} = 1 \cdot 1 = 1$$

We have oblique asymptote :

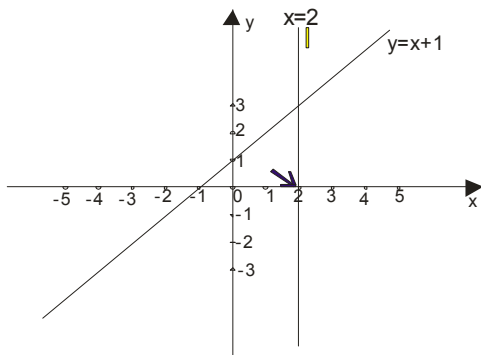
$$y = kx + n \quad \text{so:} \quad y = x + 1$$

For x=0

$$y = 0 + 1 = 1$$

For y=0

$$0 = x + 1 \rightarrow x = -1$$



x	0	-1
y	1	0

And to conclude the final graph:

