

Integrals-tasks (VII part)
Integration of some trigonometric functions

We'll give you tips for four types of integrals of trigonometric functions.

A) Integral type $\int R(\sin x, \cos x)dx$

These are integrals in which the $\sin x$ and $\cos x$ do not have degrees. Here we introduce substitution $\boxed{tg \frac{x}{2} = t}$

From substitution we get: (using the formula from trigonometry)

$$\sin x = \frac{\sin x}{1} = \frac{2 \sin \frac{x}{2} \cos \frac{x}{2}}{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}} = \frac{\cancel{\cos^2 \frac{x}{2}} \left(2 \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} \right)}{\cancel{\cos^2 \frac{x}{2}} \left(\frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} + 1 \right)} = \frac{2tg \frac{x}{2}}{tg^2 \frac{x}{2} + 1} = \frac{2t}{t^2 + 1} = \frac{2t}{1+t^2}$$

$$\cos x = \frac{\cos x}{1} = \frac{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}} = \frac{\cancel{\cos^2 \frac{x}{2}} \left(1 - \frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} \right)}{\cancel{\cos^2 \frac{x}{2}} \left(\frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} + 1 \right)} = \frac{1 - tg^2 \frac{x}{2}}{1 + tg^2 \frac{x}{2}} = \frac{1-t^2}{1+t^2}$$

As $tg \frac{x}{2} = t$ then $\frac{x}{2} = \arctgt \rightarrow x = 2\arctgt \rightarrow dx = \frac{2}{1+t^2} dt$

To summarize:

When we take substitution $\boxed{tg \frac{x}{2} = t}$ we have:

$$\boxed{\begin{aligned} \sin x &= \frac{2t}{1+t^2} \\ \cos x &= \frac{1-t^2}{1+t^2} \\ dx &= \frac{2}{1+t^2} dt \end{aligned}}$$

Substitution $tg \frac{x}{2} = t$ is *universal trigonometric substitution* and can always be used, but it's easier, depending on the look of function to use the following substitution:

B) Integral type $\int R(\operatorname{tg}x)dx$ and $\int R(\sin^2 x, \cos^2 x, \sin x \cdot \cos x)dx$

These are integrals where the function under the integral can be reduced to $\operatorname{tg}x$ or that are occurring degree of sinus and cosine and product $\sin x \cdot \cos x$

Here we introduce substitution : $\boxed{\operatorname{tg}x = t}$

From substitution we get : (using the formula from trigonometry)

$$\sin^2 x = \frac{\sin^2 x}{1} = \frac{\sin^2 x}{\sin^2 x + \cos^2 x} = \frac{\frac{\sin^2 x}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x}} = \frac{\operatorname{tg}^2 x}{\operatorname{tg}^2 x + 1} = \frac{t^2}{1 + t^2}$$

$$\cos^2 x = \frac{\cos^2 x}{1} = \frac{\cos^2 x}{\sin^2 x + \cos^2 x} = \frac{\frac{\cos^2 x}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x}} = \frac{1}{\operatorname{tg}^2 x + 1} = \frac{1}{1 + t^2}$$

$$\sin x \cdot \cos x = \frac{\sin x \cdot \cancel{\cos x}}{1} = \frac{\sin x \cdot \cos x}{\sin^2 x + \cos^2 x} = \frac{\frac{\sin x \cdot \cancel{\cos x}}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x}} = \frac{\operatorname{tg}x}{\operatorname{tg}^2 x + 1} = \frac{t}{1 + t^2}$$

$$\operatorname{tg} \frac{x}{2} = t \rightarrow \frac{x}{2} = \operatorname{arctg}t \rightarrow x = 2\operatorname{arctg}t \rightarrow dx = \frac{2}{1 + t^2} dt$$

To summarize:

When we take substitution $\boxed{\operatorname{tg}x = t}$ we have :

$$\boxed{\begin{aligned} \sin^2 x &= \frac{t^2}{1 + t^2} \\ \cos^2 x &= \frac{1}{1 + t^2} \\ \sin x \cdot \cos x &= \frac{t}{1 + t^2} \\ dx &= \frac{2}{1 + t^2} dt \end{aligned}}$$

C) Integral type $\int \sin^m x \cdot \cos^n x dx$

Two situations:

i) If m and n are integers

ii) If m and n are rational numbers

In both situations we introduce substitution $\boxed{\sin x = u}$ or $\cos x = u$ but,

i) when m and n are integers integral is reduced to integration of rational function

ii) when m and n are rational numbers we have integral of differential binomial

D) Integral type $\int \sin ax \cos bxdx; \int \sin ax \sin bxdx; \int \cos ax \cos bxdx;$

First, use the trigonometric formulas:

$$\sin ax \sin bx = \frac{1}{2} [\cos(a-b)x - \cos(a+b)x]$$

$$\sin ax \cos bx = \frac{1}{2} [\sin(a+b)x + \sin(a-b)x]$$

$$\cos ax \cos bx = \frac{1}{2} [\cos(a+b)x + \cos(a-b)x]$$

Some “tricks”:

If in integral we have expression $\sqrt{a^2 - x^2}$, it is convenient to take $x = a \sin t$ as it “destroyed” the root:

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - (a \sin t)^2} = \sqrt{a^2 - a^2 (\sin t)^2} = a \sqrt{1 - \sin^2 t} = a \cos t$$

If in integral we have expression $\sqrt{x^2 + a^2}$, it is convenient to take $x = a \tan t$ as it “destroyed” the root:

$$\sqrt{x^2 + a^2} = \sqrt{(a \tan t)^2 + a^2} = \sqrt{a^2 \tan^2 t + a^2} = a \sqrt{\tan^2 t + 1} = a \sqrt{\frac{\sin^2 t}{\cos^2 t} + 1} = a \sqrt{\frac{\sin^2 t + \cos^2 t}{\cos^2 t}} = a \sqrt{\frac{1}{\cos^2 t}} = a \frac{1}{\cos t}$$

Examples

Example 1. $\int \frac{dx}{\sin x} = ?$

This integral we have already solved in **Integrals-tasks (I part)** without using trigonometric substitution.

Now, we can use $\boxed{tg \frac{x}{2} = t}$. So :

$$\boxed{\begin{array}{l} \sin x = \frac{2t}{1+t^2} \\ dx = \frac{2}{1+t^2} dt \end{array}}$$

$$\int \frac{dx}{\sin x} = \int \frac{\frac{2}{1+t^2}}{\frac{2t}{1+t^2}} dt = \int \frac{1}{t} dt = \ln|t| + C = \ln \left| tg \frac{x}{2} \right| + C$$

Example 2. $\int \frac{2 - \sin x}{2 + \cos x} dx = ?$

Here we use $tg \frac{x}{2} = t$ because $\sin x$ and $\cos x$ have no degrees...

$$tg \frac{x}{2} = t$$

$$dx = \frac{2}{1+t^2} dt$$

$$\sin x = \frac{2t}{1+t^2}$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

$$\int \frac{2 - \sin x}{2 + \cos x} dx = \int \frac{2 - \frac{2t}{1+t^2}}{2 + \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} dt = \int \frac{2+2t^2-2t}{2+2t^2+1-t^2} \cdot \frac{2}{\cancel{1+t^2}} dt = 4 \int \frac{t^2-t+1}{(t^2+3)(1+t^2)} dt$$

This is integral of rational function ...

Beware: both terms in the denominator are irresolvable:

So:

$$\frac{t^2 - t + 1}{(t^2 + 3)(1 + t^2)} = \frac{At + B}{t^2 + 3} + \frac{Ct + D}{1 + t^2} \dots \dots \dots / \cdot (t^2 + 3)(1 + t^2)$$

$$t^2 - t + 1 = At + At^3 + B + Bt^2 + Ct^3 + 3Ct + Dt^2 + 3D$$

$$t^2 - t + 1 \equiv (A + C)t^3 + (B + D)t^2 + (A + 3C)t + B + 3D$$

$$A + C = 0, B + D = 1, A + 3C = -1 \text{ i } B + 3D = 1$$

$$A = 1/2, B = 1, C = -1/2, D = 0$$

$$4 \int \frac{t^2 - t + 1}{(t^2 + 3)(1 + t^2)} dt = 4 \int \frac{\frac{1}{2}t + 1}{t^2 + 3} dt + 4 \int \frac{\frac{-1}{2}t}{1 + t^2} dt = 2 \int \frac{t + 2}{t^2 + 3} dt - 2 \int \frac{t}{1 + t^2} dt$$

Solution is:

$$= \ln \frac{t^2 + 3}{1 + t^2} + \frac{2}{\sqrt{3}} \operatorname{arctg} \frac{t}{\sqrt{3}} + C = \boxed{\ln \frac{3 + \operatorname{tg}^2 \frac{x}{2}}{1 + \operatorname{tg}^2 \frac{x}{2}} + \frac{2}{\sqrt{3}} \operatorname{arctg} \frac{\operatorname{tg} \frac{x}{2}}{\sqrt{3}} + C}$$

Example 3. $I = \int \frac{dx}{(2 + \cos x) \sin x} = ?$

Often integral to give a letter mark, most frequently used letters : $I, J \dots$

Substitution: $\operatorname{tg} \frac{x}{2} = t$.

$$\operatorname{tg} \frac{x}{2} = t$$

$$dx = \frac{2}{1+t^2} dt$$

$$\sin x = \frac{2t}{1+t^2}$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

$$\text{So: } I = \int \frac{\cancel{2}}{\left(2 + \frac{1-t^2}{1+t^2}\right) \cdot \frac{\cancel{2}t}{1+t^2}} dt = \int \frac{dt}{\frac{2+2t^2+1-t^2}{1+t^2} \cdot t} = \int \frac{1+t^2}{(t^2+3)t} dt$$

Now, we do:

$$\frac{1+t^2}{(t^2+3)t} = \frac{A}{t} + \frac{Bt+C}{t^2+3} = \frac{At^2+3A+Bt^2+Ct^2}{t(t^2+3)}$$

$$1+t^2 = A(t^2+3) + (Bt+C)t$$

$$1+t^2 = At^2 + 3A + Bt^2 + Ct$$

$$1+t^2 = t^2(A+B) + Ct + 3A$$

$$A+B=1, \quad C=0, \quad 3A=1$$

$$A=1/3, \quad B=2/3, \quad C=0$$

$$I = \frac{1}{3} \int \frac{dt}{t} + \frac{2}{3} \int \frac{t}{t^2+3} dt = \left| \begin{array}{l} t^2+3=u \\ tdt = \frac{1}{2} du \end{array} \right| = \frac{1}{3} \ln|t| + \frac{1}{3} \int \frac{du}{u} + c$$

$$I = \frac{1}{3} \ln|t| + \frac{1}{3} \ln(t^2+3) + C = \boxed{\frac{1}{3} \ln \left| \operatorname{tg} \frac{x}{2} \right| + \frac{1}{3} \ln \left(\operatorname{tg}^2 \frac{x}{2} + 3 \right) + C}$$

Example 4. $I = \int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx = ?$

$tgx = t$

So:

$$\begin{aligned} \sin^2 x &= \frac{t^2}{1+t^2} \\ \cos^2 x &= \frac{1}{1+t^2} \\ \sin x \cdot \cos x &= \frac{t}{1+t^2} \\ dx &= \frac{2}{1+t^2} dt \end{aligned}$$

$$\begin{aligned} I &= \int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx = \int \frac{\frac{t}{1+t^2}}{\left(\frac{t^2}{1+t^2}\right)^2 + \left(\frac{1}{1+t^2}\right)^2} \cdot \frac{2}{1+t^2} dt = \int \frac{\frac{t}{1+t^2} \cdot 2}{\frac{t^4+1}{(1+t^2)^2} \cdot 1+t^2} dt = \\ &= \int \frac{2t}{t^4+1} dt = \int \frac{2t}{(t^2)^2+1} dt \left| \begin{array}{l} t^2 = z \\ 2tdt = dz \end{array} \right| = \int \frac{dz}{z^2+1} = \arctg z + C = \arctg t^2 + C = \arctg(tg^2 x) + C \end{aligned}$$

This task we could solve in another way, using trigonometric formulas

The idea is to transform the expression in the denominator. Start from the basic identity:

$$\sin^2 x + \cos^2 x = 1 \dots \dots \dots / ()^2$$

$$\sin^4 x + 2 \sin^2 x \cos^2 x + \cos^4 x = 1$$

$$\sin^4 x + \cos^4 x = 1 - 2 \sin^2 x \cos^2 x$$

$$\sin^4 x + \cos^4 x = 1 - \frac{4 \sin^2 x \cos^2 x}{2}$$

$$\sin^4 x + \cos^4 x = 1 - \frac{\sin^2 2x}{2}$$

$$\sin^4 x + \cos^4 x = \frac{2 - \sin^2 2x}{2}$$

$$\sin^4 x + \cos^4 x = \frac{1 + \boxed{1 - \sin^2 2x}}{2}$$

$$\boxed{\sin^4 x + \cos^4 x = \frac{1 + \cos^2 2x}{2}}$$

Return to the integral:

$$I = \int \frac{\sin x \cos x}{1 + \cos^2 2x} dx = \int \frac{2 \sin x \cos x}{1 + \cos^2 2x} dx = \int \frac{\sin 2x}{1 + \cos^2 2x} dx \left| \begin{array}{l} \cos 2x = t \\ -\sin 2x \cdot 2 dx = dt \\ \sin 2x dx = \frac{dt}{-2} \end{array} \right. = \frac{1}{-2} \int \frac{dt}{1+t^2}$$

$$= \frac{1}{-2} \arctgt + C = \boxed{-\frac{1}{2} \arctg(\cos 2x) + C}$$

Example 5. $\int \sqrt{a^2 - x^2} = ?$

Remember this integral?

We solved it so far in two ways: a partial integration and using Ostrogradski formula...

Here's another way of solving:

If we use substitution $x = a \sin t$

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - (a \sin t)^2} = \sqrt{a^2 - a^2 (\sin t)^2} = a \sqrt{1 - \sin^2 t} = a \cos t$$

$$x = a \sin t \rightarrow dx = a \cos t dt$$

$$\int \sqrt{a^2 - x^2} = \int a \cos t \cdot a \cos t dt = a^2 \int \cos^2 t dt = a^2 \int \frac{1 + \cos 2t}{2} dt =$$

$$\frac{a^2}{2} \int (1 + \cos 2t) dt = \frac{a^2}{2} \left(t + \frac{1}{2} \sin 2t \right) + C =$$

From substitution, we have:

$$x = a \sin t \rightarrow \sin t = \frac{x}{a} \rightarrow t = \arcsin \frac{x}{a} \quad \text{go back...}$$

$$\frac{1}{2} \sin 2t = \frac{1}{2} \cancel{\sin t \cos t} = \sin t \cdot \sqrt{1 - \sin^2 t} = \sin(\arcsin \frac{x}{a}) \sqrt{1 - \sin^2(\arcsin \frac{x}{a})} =$$

$$= \frac{x}{a} \sqrt{1 - \frac{x^2}{a^2}} = \frac{x}{a} \sqrt{\frac{a^2 - x^2}{a^2}} = \frac{x}{a^2} \sqrt{a^2 - x^2}$$

$$\text{Solution is: } \frac{a^2}{2} \left(t + \frac{1}{2} \sin 2t \right) + C = \frac{a^2}{2} \left(\arcsin \frac{x}{a} + \frac{x}{a^2} \sqrt{a^2 - x^2} \right) + C = \boxed{\frac{a^2}{2} \arcsin \frac{x}{a} + \frac{x}{2} \sqrt{a^2 - x^2} + C}$$