Integrals-tasks (VIII part)

RECURRENT FORMULA

Recurrent (recursive) formulas are formulas that depend on natural numbers.

They are used to lower the "order" an integral.

Example 1.

Determine the recursive formula for $\int x^n e^{ax} dx$ **if** $a \neq 0$ and $n \in N$

Solution:

$\int x^n e^{ax} dx = ?$

This integral will solve with partial integration (if you remember, this is integral to the first of our group).

$$I_n = \int x^n e^{ax} dx = \begin{vmatrix} x^n = u & e^{ax} dx = dv \\ nx^{n-1}\text{dx=du} & \frac{1}{a} e^{ax} = v \end{vmatrix} =$$

$$= x^n \cdot \frac{1}{a} e^{ax} - \int \frac{1}{a} e^{ax} n x^{n-1} \text{dx} = \frac{e^{ax} \cdot x^n}{a} - \frac{n}{a} \boxed{\int e^{ax} x^{n-1} \text{dx}}$$

$$= \frac{e^{ax} \cdot x^n}{a} - \frac{n}{a} \cdot I_{n-1}$$

So:

$$\boxed{I_n = \frac{e^{ax} \cdot x^n}{a} - \frac{n}{a} \cdot I_{n-1}}$$

How now use this formula?

Get a task to solve $\int x^4 e^x dx = ?$

In our formula is therefore $n = 4$ and $a = 1$.

$$I_n = \frac{e^{ax} \cdot x^n}{a} - \frac{n}{a} \cdot I_{n-1}$$

$$I_4 = \frac{e^{x} \cdot x^4}{1} - \frac{4}{1} \cdot I_{4-1} = e^x \cdot x^4 - 4I_3$$

$$\boxed{I_4 = e^x \cdot x^4 - nI_3}$$

Now, we work for *n=3,n=2,n=1*

$$I_4 = e^x \cdot x^4 - 4I_3$$
$$I_3 = e^x \cdot x^3 - 3I_2$$
$$I_2 = e^x \cdot x^2 - 2I_1$$
$$I_1 = \int e^x \cdot x dx$$

This integral we know to solve:

$$\int x e^x dx = \left| \begin{matrix} x = u & e^x dx = dv \\ dx = du & \int e^x dx = v \\ & e^x = v \end{matrix} \right| = \underbrace{x \cdot e^x}_{u \cdot v} - \underbrace{\int e^x dx}_{\int v \cdot du} = xe^x - e^x + C = \boxed{e^x(x-1)+C}$$

solution back ..

$$I_4 = e^x \cdot x^4 - 4I_3$$
$$I_3 = e^x \cdot x^3 - 3I_2$$
$$I_2 = e^x \cdot x^2 - 2I_1$$
$$I_1 = \int e^x \cdot x dx = e^x(x-1)$$

go back in I_2

$$I_2 = e^x \cdot x^2 - 2[e^x(x-1)]$$

go back in I_3

$$I_3 = e^x \cdot x^3 - 3\{e^x \cdot x^2 - 2[e^x(x-1)]\}$$

and go back in I_4

$$I_4 = e^x \cdot x^4 - 4\{e^x \cdot x^3 - 3\{e^x \cdot x^2 - 2[e^x(x-1)]\}\} + C$$

Example 2. **Determine the recursive formula for** $\int \sin^n x dx$ **if** $n \ge 2$

Solution:

This integral will solve with partial integration…

$$I_n = \int \sin^n x dx = \left| \begin{matrix} \sin^{n-1} x = u & \sin x dx = dv \\ (n-1)\sin^{n-1-1} x (\sin x)` dx = du & -\cos x = v \\ (n-1)\sin^{n-2} x \cdot \cos x dx = du & \end{matrix} \right| =$$

$$= \sin^{n-1} x \cdot (-\cos x) - \int (-\cos x)(n-1)\sin^{n-2} x \cdot \cos x dx$$

$$= -\sin^{n-1} x \cdot \cos x + (n-1)\int \sin^{n-2} x \cdot \cos^2 x dx$$

From $\sin^2 x + \cos^2 x = 1 \to \cos^2 x = 1 - \sin^2 x$, replace that instead of $\cos^2 x$

$$= -\sin^{n-1} x \cdot \cos x + (n-1)\int \sin^{n-2} x \cdot (1 - \sin^2 x) dx$$

$$= -\sin^{n-1} x \cdot \cos x + (n-1)\int (\sin^{n-2} x - \sin^n x) dx$$

$$= -\sin^{n-1} x \cdot \cos x + (n-1)\boxed{\int \sin^{n-2} x dx} - (n-1)\boxed{\int \sin^n x dx}$$

$$= -\sin^{n-1} x \cdot \cos x + (n-1) \cdot I_{n-2} - (n-1) \cdot I_n$$

Now, we have:

$$I_n = -\sin^{n-1} x \cdot \cos x + (n-1) \cdot I_{n-2} - (n-1) \cdot I_n$$

$$I_n + (n-1) \cdot I_n = -\sin^{n-1} x \cdot \cos x + (n-1) \cdot I_{n-2}$$

$$\cancel{I_n} + n \cdot I_n - \cancel{I_n} = -\sin^{n-1} x \cdot \cos x + (n-1) \cdot I_{n-2}$$

$$n \cdot I_n = -\sin^{n-1} x \cdot \cos x + (n-1) \cdot I_{n-2}$$

$$I_n = \frac{-\sin^{n-1} x \cdot \cos x + (n-1) \cdot I_{n-2}}{n}$$

$$\boxed{I_n = \frac{-\sin^{n-1} x \cdot \cos x}{n} + \frac{n-1}{n} \cdot I_{n-2}}$$

This is required recurrent formula.

We notice that if n is even number, then the gradual application of the formula obtained in the end we come to $\int dx$.

If n is odd we get $\int \sin x dx$.

Example 3. $\int \frac{dx}{\sin^n x}$ for $n \geq 2$

Solution:

Here we first use a little "trick": add $\frac{\sin x}{\sin x}$, We'll see why …

$$I_n = \int \frac{dx}{\sin^n x} = \int \frac{\sin x}{\sin x} \cdot \frac{dx}{\sin^n x} = \int \frac{\sin x dx}{\sin^{n+1} x}$$

Now do partial integration:

$$\int \frac{\sin x dx}{\sin^{n+1} x} = \left| \begin{array}{ll} u = \frac{1}{\sin^{n+1} x} & \sin x dx = dv \\ ? & -\cos x = v \end{array} \right|$$

we find this :

$$\left(\frac{1}{\sin^{n+1} x}\right)` = (\sin^{-(n+1)} x)` = -(n+1)\sin^{-(n+1)-1} x \cdot (\sin x)` = -(n+1)\sin^{-(n+2)} \cdot \cos x = -(n+1)\frac{\cos x}{\sin^{n+2}}$$

back to the task:

$$\int \frac{\sin x dx}{\sin^{n+1} x} = \left| \begin{array}{ll} \frac{1}{\sin^{n+1} x} = u & \sin x dx = dv \\ -(n+1)\frac{\cos x}{\sin^{n+2} x} dx = dv & -\cos x = v \end{array} \right| =$$

$$\frac{1}{\sin^{n+1} x}(-\cos x) - \int (-\cos x)[-(n+1)\frac{\cos x}{\sin^{n+2} x} dx] =$$

$$-\cos x \cdot \frac{1}{\sin^{n+1} x} - (n+1)\int \frac{\cos^2 x}{\sin^{n+2} x} dx =$$

$$-\cos x \cdot \frac{1}{\sin^{n+1} x} - (n+1)\int \frac{1-\sin^2 x}{\sin^{n+2} x} dx =$$

$$-\cos x \cdot \frac{1}{\sin^{n+1} x} - (n+1)[\int \frac{1}{\sin^{n+2} x} dx - \int \frac{\sin^2 x}{\sin^{n+2} x} dx] =$$

$$-\cos x \cdot \frac{1}{\sin^{n+1} x} - (n+1)[\int \frac{1}{\sin^{n+2} x} dx - \int \frac{1}{\sin^n x} dx] =$$

$$-\cos x \cdot \frac{1}{\sin^{n+1} x} - (n+1)[\int \frac{1}{\sin^{n+2} x} dx - \int \frac{1}{\sin^n x} dx] =$$

$$-\cos x \cdot \frac{1}{\sin^{n+1} x} - (n+1)[\int \frac{1}{\sin^{n+2} x} dx - \int \frac{1}{\sin^n x} dx] =$$

$$= -\cos x \cdot \frac{1}{\sin^{n+1} x} - (n+1)[I_{n+2} - I_n]$$

back to the beginning:

$$I_n = -\cos x \cdot \frac{1}{\sin^{n+1} x} - (n+1)[I_{n+2} - I_n]$$

$$I_n = -\cos x \cdot \frac{1}{\sin^{n+1} x} - (n+1)I_{n+2} + (n+1)I_n$$

$$(n+1)I_{n+2} = -\cos x \cdot \frac{1}{\sin^{n+1} x} + nI_n \cancel{+I_n} \cancel{-I_n}$$

$$(n+1)I_{n+2} = -\cos x \cdot \frac{1}{\sin^{n+1} x} + nI_n$$

$$\boxed{I_{n+2} = \frac{-\cos x}{(n+1)\cdot \sin^{n+1} x} + \frac{n}{n+1} \cdot I_n}$$

In place of n put $n-2$:.

$$I_{n+2} = \frac{-\cos x}{(n+1)\cdot \sin^{n+1} x} + \frac{n}{n+1} \cdot I_n \rightarrow\rightarrow \boxed{I_n = \frac{-\cos x}{(n-1)\cdot \sin^{n-1} x} + \frac{n-2}{n-1} \cdot I_{n-2}}$$

Example 4. Determine the recurrent formula $I_{n,m} = \int x^n \cdot \ln^m x dx$ if $n, m \in N$

$$I_{n,m} = \int x^n \cdot \ln^m x dx = \left| \begin{matrix} \ln^m x = u & x^n \text{dx=dv} \\ m \cdot \ln^{m-1} x \cdot \frac{1}{x} \text{dx=du} & \frac{x^{n+1}}{n+1} = v \end{matrix} \right| =$$

$$= \ln^m x \cdot \frac{x^{n+1}}{n+1} - \int \frac{x^{n+1}}{n+1} \cdot m \cdot \ln^{m-1} x \cdot \frac{1}{x} \text{dx}$$

$$= \ln^m x \cdot \frac{x^{n+1}}{n+1} - \int \frac{x^n \cdot \cancel{x}}{n+1} \cdot m \cdot \ln^{m-1} x \cdot \frac{1}{\cancel{x}} \text{dx}$$

$$= \ln^m x \cdot \frac{x^{n+1}}{n+1} - \frac{m}{n+1} \boxed{\int x^n \cdot \ln^{m-1} x \text{dx}}$$

$$= \ln^m x \cdot \frac{x^{n+1}}{n+1} - \frac{m}{n+1} \cdot I_{n,m-1}$$

So:

$$\boxed{I_{n,m} = \ln^m x \cdot \frac{x^{n+1}}{n+1} - \frac{m}{n+1} \cdot I_{n,m-1}}$$