Surface integrals

Surface integrals – first kind

i) If S is “part by part” smooth bilateral area given by equations:

x=x(u,v)
y=y(u,v)
z=z(u,v)

where (u,v) belongs to D and function f(x,y,z) is defined and constant on area S , then:

$$\iint_S f(x,y,z)ds = \iint_D f[x(u,v), y(u,v), z(u,v)]\sqrt{EG - F^2}\,dudv$$

$$E = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2$$

$$G = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2$$

$$F = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v}$$

ii) If the equation of area S has form z=z(x,y) , where is z=z(x,y) , then:

$$\iint_S f(x,y,z)ds = \iint_D f[x,y,z(x,y)]\sqrt{1 + p^2 + q^2}\,dxdy \qquad \text{and}$$

$$p = \frac{\partial z}{\partial x} \quad , \quad q = \frac{\partial z}{\partial y}$$

Surface integrals – first kind does not depend on ORIENTATION.

<u>Surface integrals – second kind</u>

If S is “part by part” smooth bilateral area, in which was selected one of the two parties, determined by the direction of the normal:

$\vec{n}(\cos\alpha, \cos\beta, \cos\gamma)$ and z = z(x,y) then:

$$\cos\alpha = \frac{p}{\pm\sqrt{1+p^2+q^2}}$$

$$\cos\beta = \frac{q}{\pm\sqrt{1+p^2+q^2}}$$ where is : $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$

$$\cos\gamma = \frac{-1}{\pm\sqrt{1+p^2+q^2}}$$

and P=P(x,y,z) ; Q=Q(x,y,z) ; R=R(x,y,z) three functions, defined and continuing in area S

$$\iint_S Pdydz + Qdzdx + Rdxdy = \iint_S (P\cos\alpha + Q\cos\beta + R\cos\gamma)ds$$

IMPORTANT

Can we take the + or - depending on the angle that normal build with positive part of z-line:

If angle is sharp, then it must be cos γ >0 and we are taking minus in front of root : $\cos\gamma = \frac{-1}{-\sqrt{1+p^2+q^2}}$

If angle isn`t sharp, then cos γ <0 and we are taking + in front of root: $\cos\gamma = \frac{-1}{+\sqrt{1+p^2+q^2}}$

Surface integrals – second kind depends on the orientation of the curve.

Moving to the other side of area S it changes sign.

Stokes formula:

$$\oint_L Pdx + Qdy + Rdz = \iint_S \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} dS$$

Ostrogradsky formula:

$$\iint_S (P\cos\alpha + Q\cos\beta + R\cos\gamma)dS = \iiint_V (\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z})dxdydz$$

Application:

i) **Flux of vector fields**

For vector field $\vec{A} = P(x,y,z) + Q(x,y,z) + R(x,y,z)$

$$\phi = \iint Pdydz + Qdzdx + Rdxdy = \iiint_V (\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z})dxdydz$$

div $\vec{A} = \iiint_V (\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z})dxdydz$

ii) **Circulation of vector fields**

C= $\oint Pdx + Qdy + Rdz = \iint rot\vec{A}d\vec{S}$

d$\vec{S} = \vec{n}$ ds $\qquad \vec{n} = \frac{gradu}{\pm(gradu)}$

$$\text{rot}\,\vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \iint (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})dydz + (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}) + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy$$

if rot A= 0 and div A $\neq$ 0 field is then potentially.

iii) **potential of field**

$$U(x,y,z) = \int_{x_0}^{x} P(x,y,z)dx + \int_{y_0}^{y} Q(x,y,z)dy + \int_{z_0}^{z} R(x,y,z)dz$$