

Curvilinear integral

1) curvilinear integral - first kind

i) If $f(x, y, z)$ is defined and constant function in each point of section the smooth curve c :

$$\begin{aligned} x &= x(t) \\ y &= y(t) \quad \text{where is} \quad t_1 \leq t \leq t_2, \quad \text{and} \quad ds\text{-differential arch of curve} \\ z &= z(t) \end{aligned}$$

then curvilinear integral is calculated by the formula:

$$\int_c f(x, y, z) ds = \int_{t_1}^{t_2} f[x(t), y(t), z(t)] \sqrt{(x_t')^2 + (y_t')^2 + (z_t')^2} dt$$

This integral does not depend on the orientation of the curve!

ii) If curve have form $c: y=y(x) \quad a \leq x \leq b$ then, we have:

$$\int_c f(x, y) ds = \int_a^b f(x, y(x)) \sqrt{1 + (y_x')^2} dx$$

iii) If curve have form $c: x=x(y) \quad m \leq y \leq n$ then, we have:

$$\int_c f(x, y) ds = \int_m^n f(x(y), y) \sqrt{1 + (x_y')^2} dy$$

Calculate the length of curve $c: \quad S = \int_c ds$

2. curvilinear integral – second kind

i) If the curve c is given with parametric equations:

$$\begin{aligned}x &= x(t) \\ y &= y(t) \quad \text{where is } t_0 \leq t \leq t_1 \quad \text{then:} \\ z &= z(t)\end{aligned}$$

$$\int_c P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz = \int_{t_0}^{t_1} [P(x(t), y(t), z(t))x'_t + Q(x(t), y(t), z(t))y'_t + R(x(t), y(t), z(t))z'_t]dt$$

ii) If curve have form $y=y(x)$ and $a \leq x \leq b$ then:

$$\int_c P(x, y)dx + Q(x, y)dy = \int_a^b [P(x, y(x)) + Q(x, y(x))y'_x]dx$$

iii) If curve have form $x=x(y)$ and $m \leq y \leq n$ then:

$$\int_c P(x, y)dx + Q(x, y)dy = \int_m^n [P(x(y), y)x'_y + Q(x(y), y)]dy$$

Take heed: curvilinear integral – second kind depends on the orientation of the curve: $\int_{c^+} = - \int_{c^-}$

Positive direction is contrary to the direction of movement of clockwise on the clock.

Independence curvilinear integrals of the path of integration:

Following affirmations are equivalent:

- 1) $\int_C P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz$ does not depend on the path of integration
- 2) There are function $u=u(x,y)$ so that : $du = P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz$ and:
$$\int_A^B P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = u(B) - u(A)$$
- 3) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, $\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$, $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$
- 4) $\int_C P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = 0$ if the curve c is closed.

Green's formula:

$$\oint_C P(x, y)dx + Q(x, y)dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

From the Green formula can be easily proved that the **surface of area A (D)**, which is limited with curve C,
formula:

$$A(D) = \frac{1}{2} \int_C xdy - ydx$$