

Determinant

Most said determinants are square schemes.

Determinant is a function depending on n that associates a scalar, $\det(A)$, to every $n \times n$ square matrix A .

Must be careful how marked determinants, because it is similar mark, as matrices.

By agreement, matrix is marking with square brackets, for example $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$.

Determinant is marking with $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ or $\det(A) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Finding determinants

2x2 (2 rows and 2 columns)

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Multiply the elements of the so-called main diagonal minus multiplied elements of the supporting diagonal.

Example:

$$\begin{vmatrix} 3 & 4 \\ 5 & 7 \end{vmatrix} = 3 \cdot 7 - 4 \cdot 5 = 21 - 20 = 1 \quad \begin{vmatrix} -1 & 3 \\ -5 & 12 \end{vmatrix} = (-1) \cdot 12 - (-5) \cdot 3 = -12 + 15 = 3$$

3x3 (3 rows and 3 columns)

Determinants 3×3 , we can develop in any row or column. First of all, to every element assign sign + or -

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix} \quad \text{Just to remind you: rows are } \longrightarrow \text{ , and column}$$



$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \text{If you want to develop for example, by the first row} =$$

$$= +a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = \text{Or if we do development by the second column:}$$

$$= -b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

The best is, of course, to develop by that column or the row where we have the most zeros !

Example: Calculate the value of determinant $\begin{vmatrix} 5 & 3 & 1 \\ 1 & 7 & 0 \\ 2 & 3 & 2 \end{vmatrix} = ?$

Solution:

$$\begin{vmatrix} 5 & 3 & 1 \\ 1 & 7 & 0 \\ 2 & 3 & 2 \end{vmatrix} = \text{First, above each number write the sign: } \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}, \text{ or if is easier to you, just above the numbers in}$$

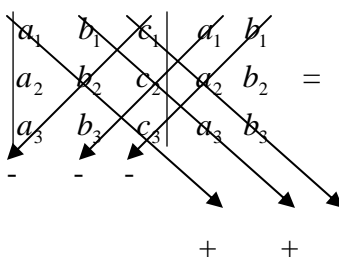
the column or row, on which you decided to expand determinant.

We have decided on the second row because it has a zero .

$$\begin{vmatrix} 5 & 3 & 1 \\ 1 & 7 & 0 \\ 2 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 5 & 3 & 1 \\ 1 & 7 & 0 \\ 2 & 3 & 2 \end{vmatrix} = -1 \begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} + 7 \begin{vmatrix} 5 & 1 \\ 2 & 2 \end{vmatrix} - 0 \begin{vmatrix} 5 & 3 \\ 2 & 3 \end{vmatrix} = -1(3 \cdot 2 - 1 \cdot 3) + 7(5 \cdot 2 - 2 \cdot 2) = -3 + 56 = 53$$

Rule of Sarrus

The sum of the products of three diagonal north-west to south-east lines of matrix elements, minus the sum of the products of three diagonal south-west to north-east lines of elements when the copies of the first two columns of the matrix are written beside it as below:



$$= a_1 b_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3 - b_1 a_2 c_3 - a_1 c_2 b_3 - c_1 b_2 a_3$$

Example: Calculate the value of determinant $\begin{vmatrix} 5 & 3 & 1 \\ 1 & 7 & 0 \\ 2 & 3 & 2 \end{vmatrix} = ?$

$$\begin{vmatrix} 5 & 3 & 1 \\ 1 & 7 & 0 \\ 2 & 3 & 2 \end{vmatrix} \begin{vmatrix} 5 & 3 \\ 1 & 7 \\ 2 & 3 \end{vmatrix} = 5 \circ 7 \circ 2 + 3 \circ 0 \circ 2 + 1 \circ 1 \circ 3 - 3 \circ 1 \circ 2 - 5 \circ 0 \circ 3 - 1 \circ 7 \circ 2 = 70 + 0 + 3 - 6 - 0 - 14 = 53$$

So, both ways, we get the same result, and you can select what is easier.

4 × 4 (4 rows and 4 columns)

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = \text{We can develop it by any row or column! Add signs : } \begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}$$

We will, for example, to develop determined by **the first column**:

$$\begin{vmatrix} + & & & \\ a_1 & b_1 & c_1 & d_1 \\ - & & & \\ a_2 & b_2 & c_2 & d_2 \\ + & & & \\ a_3 & b_3 & c_3 & d_3 \\ - & & & \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = + a_1 \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_4 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$$

Of course, now we have to develop each of the four determinants with 3 rows and 3 columns

Agree that this is not easy.

DETERMINANT CHARACTERISTICS will help us in solving problems!

DETERMINANT CHARACTERISTICS

1. Determinant changes sign if the two rows of columns change their places.
2. Value of determinants does not change if all rows and columns change their places.

$$3. \mathbf{k} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 k & b_1 k & c_1 k \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 k & b_1 & c_1 \\ a_2 k & b_2 & c_2 \\ a_3 k & b_3 & c_3 \end{vmatrix} = \mathbf{etc.}$$

$$\begin{vmatrix} a_1 & mb_1 & c_1 \\ a_2 & mb_2 & c_2 \\ a_3 & mb_3 & c_3 \end{vmatrix} = m \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$4. \begin{vmatrix} a_1 & b_1 + m_1 & c_1 \\ a_2 & b_2 + m_2 & c_2 \\ a_3 & b_3 + m_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & m_1 & c_1 \\ a_2 & m_2 & c_2 \\ a_3 & m_3 & c_3 \end{vmatrix}$$

5. If all elements of a row (column) equal to zero, the value of determinant is zero.

$$\begin{array}{c} \downarrow \\ \begin{vmatrix} 3 & 0 & 55 \\ 1 & 0 & 4 \\ 0 & 0 & 12 \end{vmatrix} = 0 \end{array} \quad \text{or} \quad \begin{array}{c} \longrightarrow \\ \begin{vmatrix} 0 & 0 & 0 & 0 \\ 77 & 68 & 34 & -80 \\ 8 & 5 & 7 & 4 \\ 4 & 5 & 9 & 8 \end{vmatrix} = 0 \end{array}$$

6. If elements of the two rows (or columns) have the same value, of determinant is zero, again.

Example:

$$\begin{array}{c} \longrightarrow \\ \longrightarrow \end{array} \begin{vmatrix} 12 & 7 & 3 \\ -9 & 4 & 6 \\ 12 & 7 & 3 \end{vmatrix} = 0$$

7. If the two species (columns) proportional to one another, the value determinants is zero .

Example:

$$\begin{array}{|c} \hline \longrightarrow \\ \hline *5 \\ \hline \longrightarrow \\ \hline \end{array} \begin{vmatrix} 2 & 3 & 4 \\ -9 & 5 & 56 \\ 10 & 15 & 20 \end{vmatrix} = 0$$

8. Value of determinant remains unchanged if the elements of a row (column) is added to the elements of some other row (column). [we can multiply that row with the same number if we need that!]

The eight properties will help us to solve determinants 4×4 , 5×5

Example1.

Calculate the value: $\begin{vmatrix} 1 & 2 & 4 & -1 \\ 2 & 2 & -3 & 5 \\ -3 & 1 & 1 & -2 \\ 4 & 2 & 1 & -2 \end{vmatrix}$

Solution:

The idea is:, using properties of the determinant No.8

$$\begin{vmatrix} \underline{1} & 2 & 4 & -1 \\ 2 & 2 & -3 & 5 \\ -3 & 1 & 1 & -2 \\ 4 & 2 & 1 & -2 \end{vmatrix}$$

↓

First, we make zero where is 4 . How? **I row * (- 4) + IV row** \longrightarrow **goes to IV row**

$$\begin{aligned} 1 \circ (-4) + 4 &= 0 \\ 2 \circ (-4) + 2 &= -6 \\ 4 \circ (-4) + 1 &= -15 \\ (-1) \circ (-4) + (-2) &= 2 \end{aligned}$$

$$\begin{vmatrix} 1 & 2 & 4 & -1 \\ 2 & 2 & -3 & 5 \\ -3 & 1 & 1 & -2 \\ 4 & 2 & 1 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 4 & -1 \\ 2 & 2 & -3 & 5 \\ -3 & 1 & 1 & -2 \\ 0 & -6 & -15 & 2 \end{vmatrix}$$

Next, we make zero where is -3.

I row *3 + III row → III row

$$\begin{aligned} 1 \circ 3 + (-3) &= 0 \\ 2 \circ 3 + 1 &= 7 \\ 4 \circ 3 + 1 &= 13 \\ (-1) \circ 3 + (-2) &= -5 \end{aligned}$$

$$\begin{vmatrix} 1 & 2 & 4 & -1 \\ 2 & 2 & -3 & 5 \\ -3 & 1 & 1 & -2 \\ 0 & -6 & -15 & 2 \end{vmatrix} \Rightarrow \begin{vmatrix} 1 & 2 & 4 & -1 \\ 2 & 2 & -3 & 5 \\ 0 & 7 & 13 & -5 \\ 0 & -6 & -15 & 2 \end{vmatrix}$$

And finally, we make zero where is 2.

(-2)*I row + II row → II row

$$\begin{aligned} (-2) \circ 1 + 2 &= 0 \\ (-2) \circ 2 + 2 &= -2 \\ (-2) \circ 4 + (-3) &= -11 \\ (-2)(-1) + 5 &= 7 \end{aligned}$$

$$\begin{vmatrix} 1 & 2 & 4 & -1 \\ 2 & 2 & -3 & 5 \\ 0 & 7 & 13 & -5 \\ 0 & -6 & -15 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 4 & -1 \\ 0 & -2 & -11 & 7 \\ 0 & 7 & 13 & -5 \\ 0 & -6 & -15 & 2 \end{vmatrix}$$

If you develop determined by the first column, we will have only one member!

$$\begin{vmatrix} 1 & 2 & 4 & -1 \\ 0 & -2 & -11 & 7 \\ 0 & 7 & 13 & -5 \\ 0 & -6 & -15 & 2 \end{vmatrix} = \begin{vmatrix} -2 & -11 & 7 \\ 7 & 13 & -5 \\ -6 & -15 & 2 \end{vmatrix} \quad \text{Rule of Sarrus ...}$$

$$\begin{vmatrix}
 -2 & -11 & 7 & -2 & -11 \\
 7 & 13 & -5 & 7 & 13 \\
 -6 & -15 & 2 & -6 & -15 \\
 - & - & - & - & -
 \end{vmatrix} = -52 - 330 - 735 + 154 + 150 + 546 = -267$$

+ + +

Of course you do not need to go step by step, but immediately take all three zero!

Example 2. Calculate
$$\begin{vmatrix}
 a & a & a & a \\
 a & b & b & b \\
 a & b & c & c \\
 a & b & c & d
 \end{vmatrix} = ?$$

Solution:

$$\begin{vmatrix}
 a & a & a & a \\
 a & b & b & b \\
 a & b & c & c \\
 a & b & c & d
 \end{vmatrix}$$

below a in the first column, we have to make all zero!

II row - I row → **II row**

III row - I row → **III row**

IV row - I row → **IV row**

$$\begin{vmatrix}
 a & a & a & a \\
 a & b & b & b \\
 a & b & c & c \\
 a & b & c & d
 \end{vmatrix} = \begin{vmatrix}
 a & a & a & a \\
 0 & b-a & b-a & b-a \\
 0 & b-a & c-a & c-a \\
 0 & b-a & c-a & d-a
 \end{vmatrix}$$

From the first row we can draw a **common** a .

$$\begin{vmatrix}
 a & a & a & a \\
 0 & b-a & b-a & b-a \\
 0 & b-a & c-a & c-a \\
 0 & b-a & c-a & d-a
 \end{vmatrix} = a \begin{vmatrix}
 1 & 1 & 1 & 1 \\
 0 & b-a & b-a & b-a \\
 0 & b-a & c-a & c-a \\
 0 & b-a & c-a & d-a
 \end{vmatrix} \longrightarrow \text{develop by 1 column}$$

$$a \begin{vmatrix}
 1 & 1 & 1 & 1 \\
 0 & b-a & b-a & b-a \\
 0 & b-a & c-a & c-a \\
 0 & b-a & c-a & d-a
 \end{vmatrix} = a \begin{vmatrix}
 b-a & b-a & b-a \\
 b-a & c-a & c-a \\
 b-a & c-a & d-a
 \end{vmatrix} = a(b-a) \begin{vmatrix}
 1 & b-a & b-a \\
 1 & c-a & c-a \\
 1 & c-a & d-a
 \end{vmatrix}$$

Go again to make the zero in the first column!

$$- \text{ II row} - \text{ I row} \longrightarrow \text{ II row} \quad : c-a-b+a=c-b$$

$$- \text{ III row} - \text{ I row} \longrightarrow \text{ III row} \quad : c-a-b+a=c-b \text{ and } d-a-b+a=d-b$$

$$a(b-a) \begin{vmatrix} 1 & b-a & b-a \\ 1 & c-a & c-a \\ 1 & c-a & d-a \end{vmatrix} = a(b-a) \begin{vmatrix} 1 & b-a & b-a \\ 0 & c-b & c-b \\ 0 & c-b & d-b \end{vmatrix} = a(b-a) \begin{vmatrix} c-b & c-b \\ c-b & d-b \end{vmatrix} = a(b-a)(c-b)(d-c)$$

So, the solution is:

$$\begin{vmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{vmatrix} = a(b-a)(c-b)(d-c)$$