

DERIVATIVES (examples - part III)

Derivatives of logarithmic functions

If we have $y = f(x)$, where is $y > 0$ and $y \neq 1$, then:

$$(\ln y)' = \frac{y'}{y} = \frac{f'(x)}{f(x)}$$

Example 1.

Find derivative of function: $y = x^x$

Solution:

$$y = x^x$$

$$\ln y = \ln x^x \quad \text{we will use rule: } \ln A^n = n \ln A$$

$$\ln y = x \ln x \quad \text{now differentiate}$$

$$\frac{y'}{y} = x' \ln x + (\ln x)' x$$

$$\frac{y'}{y} = \ln x + \frac{1}{x} x$$

$$\frac{y'}{y} = \ln x + 1 \quad \text{all multiply with } y$$

$$y' = y(\ln x + 1) \quad \text{replace } y \text{ with } x^x$$

$$y' = x^x(\ln x + 1) \quad \text{the final solution!}$$

Example 2.

Find derivative of function: $y = (\cos x)^{\sin x}$

Solution:

$$y = (\cos x)^{\sin x}$$

$$\ln y = \ln (\cos x)^{\sin x} \quad \text{rule: } \ln A^n = n \ln A$$

$$\ln y = \sin x \ln(\cos x)$$

$$\frac{y'}{y} = (\sin x)' \ln(\cos x) + [\ln(\cos x)]' \sin x \quad \text{Take heed: } \ln(\cos x) \text{ is a derivative of complex function.}$$

$$\frac{y'}{y} = \cos x \ln(\cos x) + \frac{1}{\cos x} (\cos x)' \sin x$$

$$\frac{y'}{y} = \cos x \ln(\cos x) + \frac{1}{\cos x} (-\sin x) \sin x \quad \text{simplify little...}$$

$$\frac{y'}{y} = \cos x \ln(\cos x) - \frac{\sin^2 x}{\cos x} \quad \text{all multiply with } y$$

$$y' = y \left[\cos x \ln(\cos x) - \frac{\sin^2 x}{\cos x} \right] \quad \text{replace: } y = (\cos x)^{\sin x}$$

$$y' = (\cos x)^{\sin x} \left[\cos x \ln(\cos x) - \frac{\sin^2 x}{\cos x} \right] \quad \text{is solution.}$$

Example 3.

Find derivative of function: $y = \left(\frac{\ln x}{x} \right)^{\sin x}$

Solution:

$$y = \left(\frac{\ln x}{x} \right)^{\sin x}$$

$$\ln y = \ln \left(\frac{\ln x}{x} \right)^{\sin x}$$

$$\ln y = \sin x \ln \left(\frac{\ln x}{x} \right)$$

$$\frac{y'}{y} = (\sin x)' \ln \left(\frac{\ln x}{x} \right) + \left[\ln \left(\frac{\ln x}{x} \right) \right]' \sin x$$

$$\frac{y'}{y} = \cos x \ln\left(\frac{\ln x}{x}\right) + \frac{1}{\ln x} \left(\frac{\ln x}{x}\right)' \sin x$$

$$\frac{y'}{y} = \cos x \ln\left(\frac{\ln x}{x}\right) + \frac{x}{\ln x} \frac{\left(\frac{1}{x} - \ln x\right)}{x^2} \sin x$$

$$\frac{y'}{y} = \cos x \ln\left(\frac{\ln x}{x}\right) + \frac{x}{\ln x} \frac{(1 - \ln x)}{x^2} \sin x$$

$$\frac{y'}{y} = \cos x \ln\left(\frac{\ln x}{x}\right) + \frac{(1 - \ln x)}{x \ln x} \sin x \quad \text{all multiply with } y$$

$$y' = y \left[\cos x \ln\left(\frac{\ln x}{x}\right) + \frac{(1 - \ln x)}{x \ln x} \sin x \right] \quad \text{replace: } y = \left(\frac{\ln x}{x}\right)^{\sin x}$$

$$y' = \left(\frac{\ln x}{x}\right)^{\sin x} \left[\cos x \ln\left(\frac{\ln x}{x}\right) + \frac{(1 - \ln x)}{x \ln x} \sin x \right] \quad \text{the final solution!}$$

Derivatives of function given in Parametric form

If in function $y = f(x)$ variable x and y depend on the parameters t ($x=x(t)$ and $y=y(t)$), the first derivative of function $y = f(x)$ is calculate by the formula:

$$y'_x = \frac{y'_t}{x'_t}$$

Example 1.

Calculate the first derivative function given in parametric form: $x = 2t - t^2$ and $y = 4t - t^3$

Solution:

$$\begin{aligned} x &= 2t - t^2 & \text{from here is } x'_t &= 2 - 2t \\ y &= 4t - t^3 & \text{from here is } y'_t &= 4 - 3t^2 \end{aligned}$$

Now x'_t and y'_t add to the formula:

$$y'_x = \frac{y'_t}{x'_t} = \frac{4 - 3t^2}{2 - 2t} \quad \text{and here is solution!}$$

Example 2.

Calculate the first derivative function given in parametric form: $x = r \cos t$ and $y = r \sin t$

Solution:

$$x = r \cos t \longrightarrow x'_t = -r \sin t$$

$$y = r \sin t \longrightarrow y'_t = r \cos t$$

$$y'_x = \frac{y'_t}{x'_t} = \frac{-r \cos t}{r \sin t} = \frac{-\cos t}{\sin t} = -\operatorname{ctg} t$$

Example 3.

Calculate the first derivative function given in parametric form: $x = \cos t + t \sin t$ and $y = \sin t - t \cos t$

Solution:

$$x = \cos t + t \sin t \quad \text{from here is: } x'_t = -\sin t + [t' \sin t + (\sin t)' t] = -\sin t + \sin t + t \cos t = t \cos t$$

$$y = \sin t - t \cos t \quad \text{so: } y'_t = \cos t - [t' \cos t + (\cos t)' t] = \cos t - \cos t + t \sin t = t \sin t$$

$$y'_x = \frac{y'_t}{x'_t} = \frac{t \sin t}{t \cos t} = \frac{\sin t}{\cos t} = \operatorname{tg} t$$

Derivatives of function given in Implicit form

When the function $y = f(x)$ is given in implicit form $F(x, y) = 0$, its first derivative obtained from the relation:

$$\frac{d}{dx} F(x, y) = 0$$

Example 1.

Find derivative of function: $x^3 - 2y - y^2 = 0$

Solution:

$$\text{Mark: } F(x, y) = x^3 - 2y - y^2$$

$$\frac{d}{dx} F(x, y) = 3x^2 - 2y' - 2yy' \quad \text{now this equal with 0}$$

$3x^2 - 2y' - 2yy' = 0$ from here now express y' , and that's it.

$$3x^2 = 2y' + 2yy'$$

$$3x^2 = 2y'(1+y) \quad \text{then} \quad y' = \frac{3x^2}{2(1+y)} \quad \text{is final solution.}$$

Example 2.

Find derivative of function: $x^2 + xy + y^2 + 6 = 0$

Solution:

$$F(x,y) = x^2 + xy + y^2 + 6$$

$$\frac{d}{dx} F(x,y) = 2x + y + xy' + 2yy'$$

$2x + y + xy' + 2yy' = 0$ from here we need to express y'

$$xy' + 2yy' = -2x - y$$

$$y'(x + 2y) = -2x - y \quad \longrightarrow \quad y' = \frac{-2x - y}{x + 2y} \quad \text{is final solution.}$$

Example 3.

Calculate the first derivative of function : $e^{xy} = x^3 - y^3$

Solution:

$$e^{xy} = x^3 - y^3$$

$$e^{xy}(xy)' = 3x^2 - 3y^2y'$$

$$e^{xy}(y + xy') = 3x^2 - 3y^2y'$$

$$e^{xy}y + e^{xy}xy' = 3x^2 - 3y^2y'$$

$$e^{xy}xy' + 3y^2y' = 3x^2 - e^{xy}y$$

$$y'(e^{xy}x + 3y^2) = 3x^2 - e^{xy}y \quad \text{express } y'$$

$$y' = \frac{3x^2 - e^{xy}y}{e^{xy}x + 3y^2} \quad \text{is final solution.}$$

Example 4.

Calculate the first derivative of function : $x^y - y^x = 0$

Solution:

$$x^y = y^x$$

$$\ln x^y = \ln y^x$$

$$y \ln x = x \ln y \quad \text{derivative...}$$

$$y' \ln x + y \frac{1}{x} = \ln y + \frac{1}{y} y' x$$

$$y' \ln x - \frac{x}{y} y' = \ln y - \frac{y}{x}$$

$$y' \left(\ln x - \frac{x}{y} \right) = \ln y - \frac{y}{x}$$

$$y' = \frac{\ln y - \frac{y}{x}}{\ln x - \frac{x}{y}}$$

Derivatives of inverse function

Some function f has the first derivative different from 0, and function g is its inverse function. Then function

g also has derivative and:

$$g'(x) = \frac{1}{f'(g(x))}$$

Often, this formula is written in the form of:

$$y'_x = \frac{1}{x'_y} \quad \text{or} \quad x'_y y'_x = 1$$

Example 1.

If $y = \arcsin x$, $-1 \leq x \leq 1$, $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, find derivative from y !

Solution:

Since $y = \arcsin x$, then $x = \sin y$, we use $y'_x = \frac{1}{x'_y}$ and we have :

$$\begin{aligned} (\arcsin x)' &= \frac{1}{(\sin y)'} = \frac{1}{\cos y} = \{ \text{Now use that: } \sin^2 y + \cos^2 y = 1 \text{ or } \cos y = \sqrt{1 - \sin^2 y} \} = \\ &= \frac{1}{\sqrt{1 - \sin^2 y}} = \{ \text{replace } \sin y = x \} = \frac{1}{\sqrt{1 - x^2}} \end{aligned}$$

So: $(\arcsin x)' = \frac{1}{\sqrt{1 - x^2}}$

Example 2.

If $y = \operatorname{arctg} x$ and $-\infty < x < \infty$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ find derivative from y !

Solution:

Since $y = \operatorname{arctg} x$ it must be $x = \operatorname{tg} y$, and apply $y'_x = \frac{1}{x'_y}$:

$$\begin{aligned} (\operatorname{arctg} x)' &= \frac{1}{(\operatorname{tg} y)'} = \frac{1}{\frac{1}{\cos^2 y}} = \{ \sin^2 y + \cos^2 y = 1 \} \\ &= \frac{1}{\frac{\sin^2 y + \cos^2 y}{\cos^2 y}} = \frac{1}{\frac{\sin^2 y}{\cos^2 y} + \frac{\cos^2 y}{\cos^2 y}} = \frac{1}{1 + \operatorname{tg}^2 y} \quad \text{replace: } \operatorname{tg} y = x \\ &= \frac{1}{1 + x^2} \end{aligned}$$

So: $(\operatorname{arctg} x)' = \frac{1}{1 + x^2}$

Example 3.

If $y = \log_a x$, $a > 0$, $a \neq 1$, $x > 0$ and $-\infty < y < \infty$, find derivative.

Solution:

Inverse function for $y = \log_a x$ is $x = a^y$, and by the formula $y'_x = \frac{1}{x'_y}$ is:

$$(\log_a x)' = \frac{1}{(a^y)'}$$

$$= \frac{1}{a^y \ln a} = \text{and now only replace } x = a^y$$

$$= \frac{1}{x \ln a}$$

So: $(\log_a x)' = \frac{1}{x \ln a}$