# DERIVATIVES (examples - part I )

**The primary operation in differential calculus is finding a derivative.**

**Table of derivatives**

1. C`=0
2. x`=1
3. $(x^2)$`=2x
4. $(x^n)$`=$nx^{n-1}$

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5. $(a^x)$`=$a^x$lna
6. $(e^x)$`=$e^x$
7. $(\log_a x)$`=$\frac{1}{x\ln a}$ (here is x >0 and a >0)
8. (lnx)`=$\frac{1}{x}$ (x>0)
9. $\left(\frac{1}{x}\right)^{`}=-\frac{1}{x^2}$ $(x \neq 0)$
10. $\sqrt{x}$`= $\frac{1}{2\sqrt{x}}$ (x>0)

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11. (sinx)`=cosx
12. (cosx)`= - sinx
13. (tgx)`=$\frac{1}{\cos^2 x}$ $x \neq \frac{\pi}{2} + k\pi$
14. (ctgx)`=$-\frac{1}{\sin^2 x}$ $x \neq k\pi$

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15. (arcsinx)`=$\frac{1}{\sqrt{1-x^2}}$ $|x| < 1$
16. (arccosx)`= - $\frac{1}{\sqrt{1-x^2}}$
17. (arctgx)`= $\frac{1}{1+x^2}$
18. (arcctgx)`= - $\frac{1}{1+x^2}$

**General differentiation rules:**

**1.** $[cf(x)]` = cf`(x)$ 

**2.** $[f(x) \pm g(x)]` = f`(x) \pm g`(x)$ 

(1. and 2.: **Linearity**)

**3.** $(u \circ v)` = u`v + v`u$ **Product rule**

**4.** $\left(\frac{u}{v}\right)` = \frac{u`v - v`u}{v^2}$ **Quotient rule**

**Examples:**

**1. Find derivatives for following functions:**

a) $y = x^5$
b) $y = 10^x$
c) $f(x) = \sqrt{x}$
d) $y = \log_3 x$
e) $f(x) = \sqrt[3]{x^5}$
f) $f(x) = \frac{1}{x^7}$
g) $y = \frac{1}{\sqrt[8]{x^5}}$
h) $y = x\sqrt{x}$
i) $y = \frac{x^2\sqrt{x}}{\sqrt[3]{x^2}}$

**Solution:**

a) $y = x^5 \Rightarrow y` = 5x^4$ as the 4. in table

b) $y = 10^x \Rightarrow y` = 10^x \ln 10$ as the 5. in table

c) $f(x) = \sqrt{x} \Rightarrow f`(x) = \frac{1}{2\sqrt{x}}$ as the 10. in table

d) $y = \log_3 x$ so: $y` = \frac{1}{x \ln 3}$ as the 7. in table

e) $f(x) = \sqrt[3]{x^5}$ Look out: Here first we have to" prepare" function for derivative: We will use: $\sqrt[m]{x^n} = x^{\frac{m}{n}}$. So: $\sqrt[3]{x^5} = x^{\frac{5}{3}}$ and still working as $(x^n)` = nx^{n-1}$ $\longrightarrow$ $f`(x) = \frac{5}{3}x^{\frac{5}{3}-1} = \frac{5}{3}x^{\frac{2}{3}}$

f) $f(x) = \frac{1}{x^7}$ And here we must "prepare" function. How is $\frac{1}{a^n} = a^{-n}$ it is $\frac{1}{x^7} = x^{-7}$ and

$f`(x) = -7x^{-7-1} = -7x^{-8}$

g) $y = \frac{1}{\sqrt[8]{x^5}} \longrightarrow y = x^{-\frac{5}{8}} \longrightarrow y` = -\frac{5}{8}x^{-\frac{5}{8}-1} = -\frac{5}{8}x^{-\frac{13}{8}}$

h) $y = x\sqrt{x} = x^1 x^{\frac{1}{2}} = x^{\frac{3}{2}} \longrightarrow y` = \frac{3}{2}x^{\frac{3}{2}-1} = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$

i) $y = \frac{x^2\sqrt{x}}{\sqrt[3]{x^2}} = \frac{x^2 x^{\frac{1}{2}}}{x^{\frac{2}{3}}} = \frac{x^{\frac{5}{2}}}{x^{\frac{2}{3}}} = x^{\frac{11}{6}} \longrightarrow y` = \frac{11}{6}x^{\frac{11}{6}-1} = \frac{11}{6}x^{\frac{5}{6}}$

**2. Find derivatives for following functions:**

a) $y = 5 \sin x$
b) $y = \frac{1}{2} \ln x$
c) $y = \frac{-\sqrt{3}}{4} \operatorname{tg} x$
d) $y = \pi x^3$
e) $f(x) = \frac{4}{5} \operatorname{arctg} x$
f) $f(x) = - a \operatorname{ctg} x$
g) $y = 10$
h) $y = -2abx$

**Solution:**

a) $y = 5 \sin x$ rule: **$[cf(x)]` = cf`(x)$**

$y` = 5 \cos x$

b) $y = \frac{1}{2} \ln x \longrightarrow y` = \frac{1}{2}\frac{1}{x} = \frac{1}{2x}$

c) $y = \frac{-\sqrt{3}}{4} \operatorname{tg} x \longrightarrow y` = \frac{-\sqrt{3}}{4}\frac{1}{\cos^2 x}$

d) $y = \pi x^3$ Take heed: π is also constant… $y` = \pi 3x^2$

e) $f(x) = \frac{4}{5}$ arctgx $\longrightarrow$ $f`(x)= \frac{4}{5}\frac{1}{1+x^2} = \frac{4}{5(1+x^2)}$

f) f(x) = - a ctgx $\longrightarrow$ $f`(x) = -a\,(-\frac{1}{\sin^2 x}) = \frac{a}{\sin^2 x}$

g) y = 10 $\longrightarrow$ y`=0

h) y = -2abx $\longrightarrow$ y` = -2ab

**3. Find derivatives for following functions:**

a) $y = 5x^6 - 3x^5 + 4x - 8$

b) $f(x) = 3\sin x - \frac{1}{2}e^x + 7\text{arctg}x - 5$

c) $y = \sqrt[3]{x} - \frac{2}{\sqrt{x}} + \frac{3}{x^2} - \frac{1}{5x^3} + 4$

**Solution:**

a) $y = 5x^6 - 3x^5 + 4x - 8$ use rule: **[f(x) ± g(x)]` = f `(x) ± g`(x)**

$y` = 5(x^6)` - 3(x^5)` + 4(x)` - 8`$

$y` = 30x^5 - 15x^4 + 4 - 0$

$y` = 30x^5 - 15x^4 + 4$

b) $f(x) = 3\sin x - \frac{1}{2}e^x + 7\text{arctg}x - 5$

$f`(x) = 3(\sin x)` - \frac{1}{2}(e^x)` + 7(\text{arctg}x)` - 5`$

$f`(x) = 3\cos x - \frac{1}{2}e^x + 7\frac{1}{1+x^2} - 0 = 3\cos x - \frac{1}{2}e^x + \frac{7}{1+x^2}$

c) $y = \sqrt[3]{x} - \frac{2}{\sqrt{x}} + \frac{3}{x^2} - \frac{1}{5x^3} + 4$ we have to" prepare" function for derivative…….

$y = x^{\frac{1}{3}} - 2x^{-\frac{1}{2}} + 3x^{-2} - \frac{1}{5}x^{-3} + 4$

$y` = \frac{1}{3}x^{-\frac{2}{3}} - 2(-\frac{1}{2})x^{-\frac{3}{2}} + 3(-2)x^{-3} - \frac{1}{5}(-3)x^{-4} + 0 = \frac{1}{3}x^{-\frac{2}{3}} + x^{-\frac{3}{2}} - 6x^{-3} + \frac{3}{5}x^{-4}$

**4. Find derivatives:**

a) $f(x) = x^3 \sin x$

b) $f(x) = e^x \arcsin x$

c) $y = (3x^2+1)(2x^2+3)$

d) $y = x - \sin x \cos x$

**Solution:**

As you noticing, in this task, we must use the rule for derivative products: $\mathbf{(u \circ v)` = u`v + v`u}$

a) $f(x) = x^3 \sin x$ Here is $x^3 \longleftrightarrow u$ and $\sin x \longleftrightarrow v$

$f`(x) = (x^3)` \sin x + (\sin x)` x^3$

$f`(x) = 3x^2 \sin x + \cos x\, x^3 = x^2(3\sin x + x\cos x)$

b) $f(x) = e^x \arcsin x$ $e^x \longleftrightarrow u$ and $\arcsin x \longleftrightarrow v$

$f`(x) = (e^x)`\arcsin x + (\arcsin x)` e^x$

$$f`(x) = e^x \arcsin x + \frac{1}{\sqrt{1-x^2}} e^x = e^x\left( \arcsin x + \frac{1}{\sqrt{1-x^2}}\right)$$

c) $y = (3x^2+1)(2x^2+3)$

$$y` = (3x^2+1)`(2x^2+3) + (3x^2+1)(2x^2+3)` = 6x\,(2x^2+3) + 4x\,(3x^2+1) = 2x[(6x^2+9) + (6x^2+2)] = 2x[12x^2+11]$$

d) $y = x - \sin x \cos x$

$y` = 1 - [\,(\sin x)`\cos x + (\cos x)`\sin x]$

$y` = 1 - [\,\cos x \cos x - \sin x \sin x]$ **We know that :** $\mathbf{\sin^2 x + \cos^2 x = 1}$

$y` = \sin^2 x + \cos^2 x - \cos^2 x + \sin^2 x = 2\sin^2 x$

**5. Find derivatives for following functions:**

a) $y = \frac{x^2+1}{x^2-1}$

b) $y = \frac{\cos x}{1-\sin x}$

c) $y = \frac{5-e^x}{e^x+2}$

d) $y = \frac{\ln x+1}{\ln x}$

**Solution:**

**Here we use quotient rule:** $\left(\frac{u}{v}\right)` = \frac{u`v - v`u}{v^2}$

**a)** $y = \frac{x^2+1}{x^2-1}$ here is $x^2 + 1 \longrightarrow u$ and $x^2-1 \longrightarrow v$

$y` = \frac{(x^2+1)`(x^2-1)-(x^2-1)`(x^2+1)}{(x^2-1)^2}$ **denominator shall remain so until the end!**

$y` = \frac{2x(x^2-1)-2x(x^2+1)}{(x^2-1)^2}$

$y` = \frac{2x[(x^2-1)-(x^2+1)]}{(x^2-1)^2}$ simplify little ...

$y` = \frac{-4x}{(x^2-1)^2}$ **here is the final solution!**

**b)** $y = \frac{\cos x}{1-\sin x}$

$$y` = \frac{(\cos x)`(1-\sin x) - (1-\sin x)`\cos x}{(1-\sin x)^2}$$

$$y` = \frac{-\sin x(1-\sin x) + \cos x \cos x}{(1-\sin x)^2}$$

$$y` = \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2}$$ as is: $\sin^2 x + \cos^2 x = 1$ it is:

$$y` = \frac{1-\sin x}{(1-\sin x)^2}$$

$$y` = \frac{1}{1-\sin x}$$ **final solution**

**c)** $y = \frac{5-e^x}{e^x+2}$

$$y` = \frac{(5-e^x)`(e^x+2) - (e^x+2)`(5-e^x)}{(e^x+2)^2}$$

$$y` = \frac{-e^x(e^x+2) - e^x(5-e^x)}{(e^x+2)^2}$$

$$y` = \frac{-e^x(e^x+2+5-e^x)}{(e^x+2)^2}$$ simplify little ...

$$y` = \frac{-7e^x}{(e^x+2)^2}$$

**d)** $y = \frac{\ln x + 1}{\ln x}$

$$y` = \frac{(\ln x + 1)` \ln x - (\ln x)` (\ln x + 1)}{\ln^2 x}$$

$$y` = \frac{\frac{1}{x}\ln x - \frac{1}{x}(\ln x + 1)}{\ln^2 x}$$

$$y` = \frac{\frac{1}{x}\ln x - \frac{1}{x}\ln x - \frac{1}{x}}{\ln^2 x}$$

$y` = \frac{-\frac{1}{x}}{\ln^2 x}$ **so:** $y` = \frac{-1}{x\ln^2 x}$ **is the final solution.**

6. **Determine the equation of tangent for function $y = 2x^2 - 3x + 2$ in the point A (2, y), which belongs to the function.**

**Solution:**

First, we find unknown y ; we will replace x = 2 in the function …

$y = 2 * 2^2 - 6 + 2 = 4$, and the point is actually A (2,4)

To remind you:

**T angent equation is:** $y - y_0 = f`(x_0)(x - x_0)$

$f(x) = 2x^2 - 3x + 2$ we find derivate...

$f`(x) = 4x - 3$ replace $x = 2$

$f`(2) = 8 - 3 = 5$

$y - y_0 = f`(x_0)(x - x_0)$

$y - 4 = 5(x - 2)$ simplify little ...

$y = 5x - 6$ **is requested equation of tangent**

**7. In which point on parabola $y = x^2 - 7x + 3$ is tangent line parallel with line $y = 5x + 2$?**

Solution:

$\mathbf{f(x) = x^2 - 7x + 3}$

$\mathbf{f`(x) = 2x - 7}$

Condition that two lines are parallel is $\mathbf{k_1 = k_2}$, from $\mathbf{y = 5x + 2}$ is $k = 5$, so: $f`(x) = 5$

$2x - 7 = 5$
$2x = 12$
$x = 6$

Now this value change in the equation of parabola to find y. So:

$\mathbf{y = x^2 - 7x + 3}$
$\mathbf{y = 36 - 42 + 3}$
$\mathbf{y = -3}$

**Requested point, which belongs to the parabola is: (6, -3) .**

**8. Determine the equation of normal line for function $y = x^4 - x^2 + 3$ in point M (1, y), which belongs to function .**

**Solution:**

First, find unknown coordinate y.

$y = 1 - 1 + 3 = 3$, so coordinates are M(1,3)

$\mathbf{y - y_0} = \dfrac{-1}{f`(x_0)}\mathbf{(x - x_0)}$ **equation of normal line**

$\mathbf{y = x^4 - x^2 + 3}$

$\mathbf{y` = 4x^3 - 2x}$

$y`(1) = 4 - 2 = 2$

$y - 3 = \dfrac{-1}{2}(x - 1)$ simplify little ...

$\mathbf{2y - 6 = -x + 1} \longrightarrow$ **n : $x + 2y - 7 = 0$ is requested solution**