

Binomial formula

Acquainted yourself first with a character : $n!$ → where "!" is the factorial operator.

$$n! = n \circ (n-1) \circ (n-2) \circ \dots \circ 3 \circ 2 \circ 1$$

Example:

$$5! = 5 \circ 4 \circ 3 \circ 2 \circ 1 = 120 \quad \text{or} \quad 7! = 7 \circ 6 \circ 5 \circ 4 \circ 3 \circ 2 \circ 1 = 5040$$

By definition, is $0! = 1$

In tasks we often use to separate the factorial as a product of few members and new factorial.

For example:

$$(n+2)! = (n+2)(n+1)n(n-1) \circ \dots \circ 2 \circ 1$$

$$(n+2)! = (n+2)(n+1)n!$$

or

$$(n+2)! = (n+2)(n+1)n(n-1)! \quad \text{or} \quad \dots$$

Example 1. Reduce the fraction:

$$\frac{(n-1)!}{(n-3)!}$$

Solution:
$$\frac{(n-1)!}{(n-3)!} = \frac{(n-1)(n-2)(n-3)!}{(n-3)!} = (n-1)(n-2)$$

Example 2. Solve the equation:

$$\frac{(2x)!}{(2x-3)!} = \frac{20x!}{(x-2)!}$$

Solution:

$$\frac{(2x)!}{(2x-3)!} = \frac{20x!}{(x-2)!}$$

$$\frac{(2x)(2x-1)(2x-2)(2x-3)!}{(2x-3)!} = \frac{20x(x-1)(x-2)!}{(x-2)!}$$

$$(2x)(2x-1)(2x-2) = 20x(x-1)$$

$$2x(2x-1)2(x-1) = 20x(x-1)$$

$$2x-1 = 5 \quad \text{and from here is} \quad x=3$$

$\binom{n}{k}$ -is interpreted as the number of k -element subsets of an n -element set, that is the number of ways that k things can be "chosen" from a set of n things.

$\binom{n}{k}$ is often read as " **n choose k** " and called the **choose function** of n and k .

$$\binom{n}{k} = \frac{n(n-1)(n-2)\dots(n-k+1)}{k!}$$

Examples: $\binom{10}{2} = \frac{10 \circ 9}{2 \circ 1} = 45$, $\binom{15}{3} = \frac{15 \circ 14 \circ 13}{3 \circ 2 \circ 1} = 455$

To have **the speed in the work** we have to remember that:

$$\binom{n}{0} = 1 \quad \text{For example : } \binom{5}{0} = 1 \quad \binom{12}{0} = 1 \quad \dots$$

$$\binom{n}{n} = 1 \quad \text{For example : } \binom{7}{7} = 1 \quad \binom{100}{100} = 1 \quad \dots$$

$$\binom{n}{1} = \binom{n}{n-1} = n \quad \text{For example: } \binom{4}{1} = \binom{4}{3} = 4 \quad \binom{50}{1} = \binom{50}{49} = 50$$

And most important: $\binom{n}{k} = \binom{n}{n-k}$

For example, we get to decide $\binom{20}{18}$. With this policy we resolve:

$$\binom{20}{18} = \binom{20}{2} = \frac{20 \circ 19}{2 \circ 1} = 190. \quad \text{It is easier!}$$

Now we can see how it seems **binomial form**:

$$(a+b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \dots + \binom{n}{n-1} a^1 b^{n-1} + \binom{n}{n} a^0 b^n$$

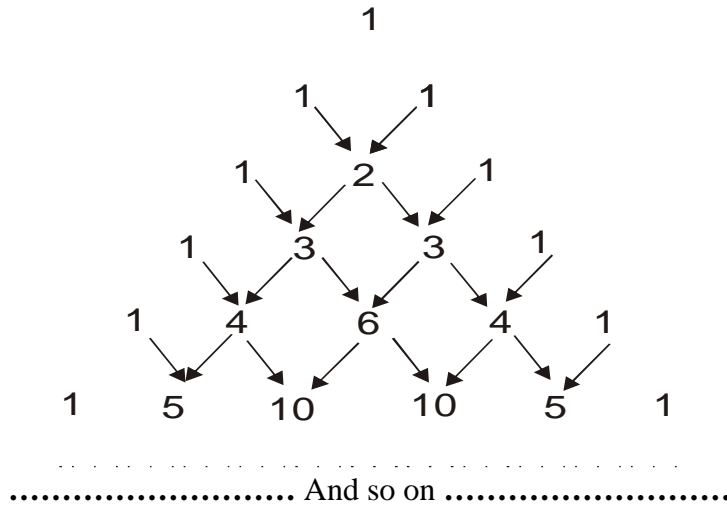
This formula can be easily proved by the application of mathematical induction.

What is important to notice?

- The development has always $n + 1$ members
- **a** begins with the degree **n**, and any member of the following decreases until it comes to **zero**, while **b** starts from **zero**, and each member of the next growing until come to the degree **n**
- $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n-1}, \binom{n}{n}$ are the **binomial coefficients**, and for them we have one interesting thing:

Paskal triangle

$(a+b)^0 = 1$	coefficient 1	1
$(a+b)^1 = a+b$	coefficients are 1 and 1	1 1
$(a+b)^2 = a^2+2ab+b^2$	coefficients are 1, 2, 1	1 2 1
$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$	coefficients are 1, 3, 3, 1	1 3 3 1
$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$	coefficients are 1, 4, 6, 4, 1 etc..	1 4 6 4 1



General (any) member in the form of a developed stage is required by the formula:

$$T_{k+1} = \binom{n}{k} a^{n-k} b^k$$

EXAMPLES:

1) $(3 + 2x)^5 = ?$

$$(3 + 2x)^5 = [\text{Here is } a = 3, b = 2x \text{ and } n = 5]$$

$$\binom{5}{0} 3^5 (2x)^0 + \binom{5}{1} 3^4 (2x)^1 + \binom{5}{2} 3^3 (2x)^2 + \binom{5}{3} 3^2 (2x)^3 + \binom{5}{4} 3^1 (2x)^4 + \binom{5}{5} 3^0 (2x)^5$$

it's easier to extract the binomial coefficients, and fix them first:

$$\binom{5}{0} = \binom{5}{5} = 1$$

$$\binom{5}{1} = \binom{5}{4} = 5$$

$$\binom{5}{2} = \frac{5 \cdot 4}{2 \cdot 1} = 10 = \binom{5}{3}$$

$$\begin{aligned} &= 1 \cdot 3^5 \cdot 1 + 5 \cdot 3^4 \cdot 2 \cdot x + 10 \cdot 3^3 \cdot 2^2 \cdot x^2 + 10 \cdot 3^2 \cdot 2^3 \cdot x^3 + 5 \cdot 3 \cdot 2^4 \cdot x^4 + 1 \cdot 1 \cdot 2^5 \cdot x^5 = \\ &= 243 + 810x + 1080x^2 + 720x^3 + 240x^4 + 32x^5 \end{aligned}$$

2) $(1 + i)^6 = ?$

$$(1 + i)^6 = [\text{Here is } a = 1, b = i \text{ and } n = 6]$$

$$\binom{6}{0} 1^6 \cdot i^0 + \binom{6}{1} 1^5 \cdot i^1 + \binom{6}{2} 1^4 \cdot i^2 + \binom{6}{3} 1^3 \cdot i^3 + \binom{6}{4} 1^2 \cdot i^4 + \binom{6}{5} 1^1 \cdot i^5 + \binom{6}{6} 1^0 \cdot i^6$$

$$\binom{6}{0} = \binom{6}{6} = 1$$

$$\binom{6}{1} = \binom{6}{5} = 6$$

$$\binom{6}{2} = \frac{6 \cdot 5}{2 \cdot 1} = 15 = \binom{6}{4}$$

$$\binom{6}{3} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20$$

To remind you:

$$\left. \begin{array}{l} i^1 = i \\ i^2 = -1 \\ i^3 = -i \\ i^4 = 1 \end{array} \right\} \text{So: } \begin{array}{l} i^5 = i^4 \cdot i = i \\ i^6 = i^4 \cdot i^2 = -1 \end{array}$$

Let's go back to the task:

$$\begin{aligned} &= 1 \cdot 1 \cdot 1 + 6 \cdot 1 \cdot i + 15 \cdot 1 \cdot (-1) + 20 \cdot 1 \cdot (-i) + 15 \cdot 1 \cdot 1 + 6 \cdot i + 1 \cdot 1 \cdot (-1) \\ &= 1 + 6i - 15 - 20i + 15 + 6i - 1 \\ &= -8i \end{aligned}$$

$$\text{So: } (1+i)^6 = -8i$$

3) Determine the **fifth member** in the form of a developed stage: $\left(x^{\frac{1}{2}} + x^{\frac{2}{3}}\right)^{12}$

Solution:

$$a = x^{\frac{1}{2}}, \quad b = x^{\frac{2}{3}}, \quad n = 12$$

We will use formula:

$$T_{k+1} = \binom{n}{k} a^{n-k} b^k$$

As the search is for the fifth member:

$$\begin{aligned} T_5 &= T_{4+1} \\ &= \binom{12}{4} \left(x^{\frac{1}{2}}\right)^{12-4} \left(x^{\frac{2}{3}}\right)^4 \\ &= \binom{12}{4} x^4 \cdot x^{\frac{8}{3}} \\ &= \frac{12 \cdot 11 \cdot 10 \cdot 9}{4 \cdot 3 \cdot 2 \cdot 1} x^{4+\frac{8}{3}} \\ &= 495 \cdot x^{\frac{20}{3}} \end{aligned}$$

4) Determine member that does not include x in developed form of $(x + x^{-2})^{12}$

Solution:

$$a = x, \quad b = x^{-2}, \quad n = 12$$

We will use formula $T_{k+1} = \binom{n}{k} a^{n-k} b^k$ to find k

$$\begin{aligned} T_{K+1} &= \binom{n}{k} a^{n-k} \cdot b^k \\ &= \binom{12}{k} x^{12-k} (x^{-2})^k \\ &= \binom{12}{k} x^{12-k} \cdot x^{-2k} \\ &= \binom{12}{k} x^{12-3k} \end{aligned}$$

Since we need a member **that** does not contain **x**, carried out by comparison:

$$\begin{aligned} x^{12-3k} &= x^0 \\ 12 - 3k &= 0 \\ 3k &= 12 \\ k &= 4 \end{aligned}$$

So, in question was ($T_{4+1} = T_5$) the fifth member.

5) The sum of coefficient of the first, second and third member in the development stage of $\left(x^2 + \frac{1}{x}\right)^n$ is 46.

Find a member that does not contain x.

Solution:

$$\begin{aligned} \binom{n}{0} + \binom{n}{1} + \binom{n}{2} &= 46 \\ 1 + n + \frac{n(n-1)}{2} &= 46 \\ 2 + 2n + n^2 - n &= 92 \\ n^2 + n - 90 &= 0 \\ n_{1,2} &= \frac{-1 \pm 9}{2} \\ n &= 9 \end{aligned}$$

So: $a = x^2$, $b = \frac{1}{x}$, $n = 9$

$$\begin{aligned}
 T_{K+1} &= \binom{n}{k} a^{n-K} \cdot b^K \\
 &= \binom{9}{k} (x^2)^{9-k} \left(\frac{1}{x}\right)^k \\
 &= \binom{12}{k} x^{18-k} \cdot x^k \\
 &= \binom{12}{k} x^{18-3k}
 \end{aligned}$$

$$\begin{aligned}
 x^{18-3k} &= x^0 \\
 18-3k &= 0 \\
 3k &= 18 \\
 k &= 6
 \end{aligned}$$

Means that there is a seventh member.

6) Find the coefficients with x^3 in developed stage $\left(\frac{1}{4x} - 2x^2\right)^{12}$

Solution:

$$a = \frac{1}{4x}, b = -2x^2, n = 12$$

$$\begin{aligned}
 T_{K+1} &= \binom{n}{k} a^{n-K} \cdot b^K \\
 &= \binom{12}{k} \left(\frac{1}{4x}\right)^{12-k} \cdot (-2x^2)^k \\
 &= \binom{12}{k} \left(\frac{1}{4}\right)^{12-k} x^{k-12} \cdot (-2)^k \cdot x^{2k} \\
 &= \binom{12}{k} \left(\frac{1}{4}\right)^{12-k} \cdot (-2)^k \cdot \underbrace{x^{3k-12}}_{x^3}
 \end{aligned}$$

So:

$$\begin{aligned}
 x^{3k-12} &= x^3 \\
 3k-12 &= 3 \\
 3k &= 15 \\
 k &= 5
 \end{aligned}$$

And the coefficients with x^3 will be:

$$\begin{aligned} \binom{12}{k} \left(\frac{1}{4}\right)^{12-k} (-2)^k &= \\ \binom{12}{5} \left(\frac{1}{4}\right)^7 \cdot (-2)^5 &= \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \cdot \frac{1}{4^7} \cdot (-32) \\ &= -\frac{99}{64} = -1,546875 \end{aligned}$$

7) We have $\binom{n}{1} : \binom{n}{2} = 2 : 11$ and $\left(\frac{x}{\sqrt{4}} + \frac{\sqrt{y}}{x}\right)^n$. Determine the fifth member.

Solution:

$$\begin{aligned} \binom{n}{1} : \binom{n}{2} &= 2 : 11 \\ n : \frac{n(n-1)}{2} &= 2 : 11 \\ 11n &= n(n-1) \\ 11n &= n^2 - n \\ n^2 - 12n &= 0 \\ n(n-12) = 0 &\Rightarrow n = 0 \quad \vee \quad n = 12 \rightarrow \text{can not} \end{aligned}$$

Because $a = \frac{x}{\sqrt{4}}$, $b = \frac{\sqrt{y}}{x}$, $n = 12$ and we must find the fifth member:

$$\begin{aligned} T_{k+1} &= \binom{n}{k} a^{n-k} \cdot b^k \\ T_{k+1} &= \binom{n}{k} a^{n-k} \cdot b^k \\ T_5 = T_{4+1} &= \binom{12}{4} \left(\frac{x}{\sqrt{y}}\right)^8 \cdot \left(\frac{\sqrt{y}}{x}\right)^4 \\ &= \binom{12}{4} \frac{x^8}{y^4} \cdot \frac{y^2}{x^4} \\ &= \frac{12 \cdot 11 \cdot 10 \cdot 9}{4 \cdot 3 \cdot 2 \cdot 1} \cdot \frac{x^4}{y^2} \\ &= 495x^4y^{-2} \end{aligned}$$

8) At the railway station should be received in the same direction n people. On how many possible ways, considering the time of arrival, they can arrive at the station?

Solution:

Think:

- Can all come at different time
- To arrive two together, the other in a different time
- To arrive three together, the other in a different time
- Etc.
- To arrive in groups of 2
- To arrive in groups of 3
- itd - Etc.

Number of all the possibilities:

$$C_1^n + C_2^n + C_3^n + \dots + C_n^n = \\ \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \dots + \binom{n}{n} =$$

To calculate this, let us start from binomial formula :

$$(a+b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \dots + \binom{n}{n} a^0 b^n$$

If instead of **a** and **b** put **1**, you will receive:

$$(1+1)^n = \binom{n}{0} \cdot 1 \cdot 1 + \binom{n}{1} \cdot 1 \cdot 1 + \dots + \binom{n}{n} \cdot 1 \cdot 1$$

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n}$$

$$\binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n - \binom{n}{0}$$

$$\binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n - 1$$

Number of possibilities : $2^n - 1$